Linear Waves, Instabilities and Energy Principle

→ Contents
- this unit presents the linear structure, response theory and energetics for MHD

- proceed by:
  a.) linear waves
  b.) Least Action and Energy Principle
  c.) simple linear instabilities

- Later discuss nonlinear evolution, i.e.:
  i.e. a.) MHD shocks
  b.) collisionless shocks
  c.) MHD turbulence (later).

A.) Linear Waves in MHD

i.) Simple Cases

- before proceeding with full crank useful to discuss some limiting cases in depth
- always have $\underline{B}_0 = B_0 \hat{z}$, $\rho = \rho_0$, $P = P_0$ } uniform

- consider

	$\nabla \cdot \underline{V} = 0$	$\nabla \cdot \underline{V} \neq 0$	
$\underline{k} = k\hat{z}$	Shear Alfven	Acoustic	– parallel propagation
$\underline{k} = k\hat{x}$	X	Magnetosonic	– perpendicular propagation

$\rightarrow$ $\underline{k} = k\hat{z}$, $\underline{\nabla}\cdot\underline{V} = 0$ Shear Alfven Wave

$$\left.\begin{aligned} \rho_0 \frac{\partial \underline{\tilde{V}}}{\partial t} &= -\nabla\left(\tilde{p} + \frac{\widetilde{B^2}}{8\pi}\right) + \frac{\underline{B_0}\cdot\underline{\nabla}\,\underline{\tilde{B}}}{4\pi} \\ \frac{\partial \underline{\tilde{B}}}{\partial t} &= \underline{B_0}\cdot\underline{\nabla}\,\underline{\tilde{V}} \end{aligned}\right\}$$ linearized eqns.

Now, $\underline{\nabla}\cdot\underline{\tilde{V}} = 0 \Rightarrow$

$$-\nabla^2\left(\tilde{p} + \frac{\underline{B_0}\cdot\underline{\tilde{B}}}{8\pi}\right) + B_0\cdot\nabla\cancel{(\nabla\cdot\underline{\tilde{B}})}^{\,0} = 0$$

$\left\{\begin{array}{l}\rho_0, B_0 \\ \text{uniform}\end{array}\right.$

$$\therefore \quad \tilde{p} + \frac{\underline{B_0}\cdot\widetilde{\underline{B}}}{8\pi} = 0$$

$\rightarrow$ "perturbed pressure balance"

$\rightarrow$ holds for incompressible (and weakly compressible) modes

$$\Rightarrow \quad \rho_0 \frac{\partial \underline{\tilde{V}}}{\partial t} = \frac{B_0}{4\pi}\frac{\partial}{\partial z}\underline{\tilde{B}}$$

$$\frac{\partial \underline{\tilde{B}}}{\partial t} = B_0 \frac{\partial}{\partial z}\underline{\tilde{V}}$$

$$\therefore \quad \boxed{\frac{\partial^2 \underline{\tilde{V}}}{\partial t^2} = \frac{B_0^2}{4\pi\rho_0}\frac{\partial^2}{\partial z^2}\underline{\tilde{V}}}$$

$B_0^2/4\pi\rho_0 = V_A^2$ Alfven velocity

$\Rightarrow \begin{cases} \omega^2 = k_{\parallel}^2 V_A^2 & \rightarrow \text{dispersion relation for } \underline{\text{shear}}\ \underline{\text{Alfven}}\ \underline{\text{wave}} \\ V_{ph} = V_{gr} = V_A \hat{z} & \rightarrow \text{speed} \begin{cases} \text{phase} \\ \text{group} \end{cases} \end{cases}$

wave propagates along $\hat{z}$ at Alfven speed

→ wave is consequence of <u>magnetic</u> <u>tension</u>

$\frac{T}{\mu} \rightarrow \frac{B/4\pi}{\rho_0/B}$ ~ tension-in line $\Rightarrow V_A^2$

↳ mass-per-line

→ tension ↔ plucking $\Rightarrow \underline{V} \perp B_0$

$\left(\begin{array}{c} \nabla \cdot \underline{V} = 0 \\ \text{parallel variation} \end{array} \right)$

i.e. $\begin{cases} \underline{V} = \tilde{V}_x \hat{x} \\ \underline{B} = \frac{\partial}{\partial z}(\tilde{V}_x B_0) = \tilde{B}_x \hat{x} \end{cases}$

in shear Alfven wave:

$\begin{cases} \underline{V},\ \underline{B} \perp \underline{B_0} \\ \underline{V} \parallel \underline{B}, \text{ but out of phase} \end{cases}$

→ energetics → construct "Poynting theorem"

$$\rho_0 \frac{\partial \tilde{V}}{\partial t} = \frac{B_0}{4\pi} \frac{\partial}{\partial z} \tilde{B} \qquad (1)$$

$$\frac{\partial \tilde{B}}{\partial t} = B_0 \frac{\partial}{\partial z} \tilde{V} \qquad (2)$$

∴ construct energy evolution

$$\mathcal{E} = \frac{\rho \tilde{V}^2}{2} + \frac{\tilde{B}^2}{8\pi} \rightarrow \text{energy density}$$

∴ $(1) \cdot \underline{V}$ and $(2) \cdot \underline{B}$ ⇒

$$\frac{\partial}{\partial t}\left(\frac{\rho_0 \tilde{V}^2}{2} + \frac{\tilde{B}^2}{8\pi}\right) = \frac{B_0}{4\pi}\left(\tilde{V} \cdot \frac{\partial \tilde{B}}{\partial z} + \tilde{B} \cdot \frac{\partial V}{\partial z}\right)$$

∴

$$\frac{\partial}{\partial t}\left(\frac{\rho_0 \tilde{V}^2}{2} + \frac{\tilde{B}^2}{8\pi}\right) = \frac{B_0}{4\pi} \frac{\partial}{\partial z} (\tilde{V} \cdot \tilde{B})$$

and have Poynting form: $\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \underline{S} = 0$

$$\underline{S} = -\frac{B_0}{4\pi} (\underline{V} \cdot \underline{B}) \rightarrow \text{wave energy density flux}$$

N.B. $\underline{S} = \frac{c}{4\pi} \underline{E} \times \underline{B}$, $\underline{p} = \underline{S}/c^2$

δ wave energy density flux ↳ wave momentum density

$\underline{E} = -\frac{\underline{v} \times \underline{B}_0}{c}$

$$\underline{S} = -\frac{1}{4\pi} (\tilde{\underline{v}} \times \underline{B}_0) \times \tilde{\underline{B}} = \frac{1}{4\pi}\left[(\cancel{\tilde{\underline{B}} \cdot \underline{B}_0}) \tilde{\underline{v}} - (\underline{v} \cdot \underline{B}) \underline{B}_0 \right]$$

$$= -\frac{\underline{B}_0}{4\pi} (\underline{v} \cdot \underline{B})$$

$$\underline{S} = -\frac{\underline{B}_0}{4\pi} \underline{v} \cdot \underline{B}$$

i.e. – energy flows along field

– $\underline{S} \sim \underline{v} \cdot \underline{B}$

$H_c = \int d^3x \; \underline{v} \cdot \underline{B}$ → cross helicity
→ conserved in ideal MHD

Ex.: Show H_c conserved.

→ another way to formulate shear Alfven wave

since $\tilde{\underline{v}} \perp B_0$ write $\underline{v} = \nabla\phi \times \hat{z}$ ↳ velocity potential
$\tilde{\underline{B}} \perp B_0$ $\underline{B} = \nabla A \times \hat{z}$
↳ magnetic potential

i.e $\underline{E} = \underline{E}_\perp$ so $\underline{v} = \frac{c}{B_0^2} \underline{E} \times \underline{B}_0$ in shear Alfven now, $\displaystyle \frac{\partial \underline{v}}{\partial t} = -\frac{1}{\rho_0} \underline{\nabla}\left(\widetilde{p+\frac{B^2}{8\pi}}\right) + \frac{\underline{B_0}\cdot\underline{\nabla}\,\underline{B}}{4\pi\rho_0}$

as $\tilde{\underline{v}}, \tilde{\underline{B}} \perp B_0$, take $\hat{z}\cdot\underline{\nabla}\times$ $\Rightarrow$

$$\hat{z}\cdot\frac{\partial \underline{\omega}}{\partial t} = 0 + \frac{B_0}{4\pi\rho_0}\frac{\partial}{\partial z}\,\hat{z}\cdot(\underline{\nabla}\times\underline{B})$$

Now, $\underline{v} = \underline{\nabla}\phi\times\hat{z}$

$= (\partial_y\phi, -\partial_x\phi, 0)$

$\underline{\omega}_z = \hat{z}\cdot\omega = -\nabla_\perp^2\phi$

$\hat{z}\cdot\underline{\nabla}\times\underline{\tilde{B}} = \frac{4\pi}{c}\tilde{J}_z$

$\underline{\nabla}(\underline{\nabla}\cdot\underline{A}) - \nabla^2 A = +\frac{4\pi}{c}\tilde{J}_z$

$\Rightarrow$

$$\underbrace{\frac{\partial}{\partial t}\nabla_\perp^2\phi}_{\text{vorticity evolution}} = \underbrace{\frac{B_0}{4\pi\rho_0}\frac{\partial}{\partial z}\nabla_\perp^2 A}_{\text{magnetic torque}}$$

$\underline{\nabla}\times(\underline{J}\times\underline{B})$

and $\displaystyle \frac{\partial \underline{\tilde{B}}}{\partial t} = B_0\frac{\partial}{\partial z}\underline{v}$ and $\hat{z}\cdot\underline{\nabla}\times$ $\Rightarrow$

$$\underbrace{\frac{\partial}{\partial t}\nabla_\perp^2 A}_{\text{current evolution}} = B_0\underbrace{\frac{\partial}{\partial z}\nabla_\perp^2\phi}_{\parallel \text{ vorticity gradient}}$$

observe if "un-$\nabla_\perp^2$", have:

$$\frac{\partial A}{\partial t} - B_0 \frac{\partial}{\partial z}\phi = 0$$

$\Rightarrow$ basically means $E_{\parallel} = 0$ for Alfven waves.

$$\underline{E} = -\frac{\underline{v} \times \underline{B}_0}{c} \quad \therefore \quad \hat{z} \cdot \frac{\tilde{\underline{v}} \times B_0 \hat{z}}{c} = 0 \quad \checkmark$$

$\therefore$ can write shear Alfven wave equations as

$$\left.\begin{aligned} E_{\parallel} = 0 = \frac{\partial A}{\partial t} - B_0 \frac{\partial \phi}{\partial z} = 0 \\ \frac{\partial}{\partial t} \nabla_\perp^2 \phi = \frac{B_0}{4\pi\rho_0} \frac{\partial}{\partial z} \nabla_\perp^2 A \end{aligned}\right\}$$

$\Rightarrow$ example of 'reduced equations'.

Now, need also consider:

$\rightarrow$ $\underline{k} = k\hat{z}$, $\quad \underline{\nabla} \cdot \underline{v} \neq 0$

What happens?

Now, $$\frac{\partial \underline{V}}{\partial t} = -\left(\frac{1}{\rho_0}\right)\underline{\nabla}\left(\tilde{p} + \frac{\underline{B}_0\cdot\tilde{\underline{B}}}{4\pi}\right) + \frac{\underline{B}_0\cdot\underline{\nabla}\,\underline{B}}{4\pi\rho_0}$$

$$\frac{\partial \tilde{\underline{B}}}{\partial t} = \underline{B}_0\cdot\underline{\nabla}\,\underline{V} - B_0\,\underline{\nabla}\cdot\tilde{\underline{V}}$$

$$\underline{k} = k\hat{z} \qquad \underline{\nabla}\cdot\underline{V} \neq 0$$

$$\Rightarrow \frac{\partial \tilde{V}_z}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{\tilde{p}}{\rho_0}\right) - \cancel{\frac{\partial}{\partial z}\left(\frac{B_0\cdot\tilde{B}}{4\pi\rho_0}\right)} + \cancel{B_0\frac{\partial}{\partial z}\left(\frac{\tilde{B}_z}{4\pi\rho_0}\right)}$$

and $$\frac{\partial \tilde{B}_z}{\partial t} = \cancel{B_0\,\partial_z\tilde{V}_z} - \cancel{B_0\,\partial_z\tilde{V}_z}$$

$\therefore$ all that's left is simple acoustic mode

$$\frac{\partial \tilde{V}_z}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{\tilde{p}}{\rho_0}\right)$$

$$\frac{\tilde{p}}{p_0} = \gamma\frac{\tilde{\rho}}{\rho_0} \qquad \text{from } p = p_0(\rho/\rho_0)^\gamma$$

$$\frac{\partial\tilde{\rho}}{\partial t} = -\rho_0\,\underline{\nabla}\cdot\tilde{\underline{V}} = -\rho_0\,\partial_z\tilde{V}_z$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\tilde{\rho} = \gamma\frac{p_0}{\rho_0}\frac{\partial^2}{\partial z^2}\tilde{\rho}$$

$$\Rightarrow \quad \omega^2 = c_s^2 k_z^2 \quad , \quad c_s^2 = \gamma \, \underbrace{P}_{\text{"stiffness"}} / \rho_0$$

energy density

$\rightarrow \underline{k} = k \, \hat{x}$ — Perpendicular Propagation

Now $\underline{B} = B_0 \hat{z}$, so

$\rightarrow \underline{k} = k\,\hat{x}$ <u>must</u> compress magnetic field

$\therefore$

$\rightarrow$ no incompressible cross-field propagation is possible

Now

$$\frac{\partial \underline{V}}{\partial t} = -\frac{\underline{\nabla}}{\rho_0}\widetilde{\left(P + \frac{B^2}{8\pi}\right)} + \cancel{\frac{B_0 \cdot \underline{\nabla}\, \underline{\tilde{B}}}{4\pi\rho_0}}$$

and

$$\frac{\partial}{\partial t} \underline{B}/\rho = \cancel{\frac{B_0 \cdot \underline{\nabla}\, \underline{\tilde{V}}}{\rho_0}} \quad \text{— freezing in}$$

so can take short-cut via:

$$\frac{d}{dt} B/\rho = 0 \quad \Rightarrow \quad \tilde{B} = B_0 \frac{\tilde{\rho}}{\rho_0}$$

$$\frac{\partial \underline{V}}{\partial t} = -\frac{1}{\rho_0} \nabla \left(\overset{\text{thermal}}{P_T} + \underset{\text{magnetic}}{P_B} \right)$$

$$P_T = P_0 \left(\rho/\rho_0\right)^\gamma, \qquad \tilde{P}_T = \gamma P_0 \left(\tilde{\rho}/\rho_0\right)$$

$$P_B = B^2/8\pi, \qquad \tilde{P}_B = 2\frac{B_0^2}{8\pi}\left(\tilde{\rho}/\rho_0\right)$$

(i.e. "γ_{eff}" $= 2$, for field)

$$\frac{\partial}{\partial t}\left(\underline{\nabla}\cdot\underline{\tilde{V}}\right) = -\nabla^2\left[\frac{\gamma P_0}{\rho_0} + \frac{2 B_0^2}{8\pi\rho_0}\right]\frac{\tilde{\rho}}{\rho_0}$$

but $\underline{\nabla}\cdot\underline{V} = -\frac{\partial}{\partial t}\frac{\tilde{\rho}}{\rho_0}$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2}\left(\tilde{\rho}/\rho_0\right) = \nabla^2\left[\frac{\gamma P_0}{\rho_0} + \frac{2 B_0}{8\pi\rho_0}\right]\left(\tilde{\rho}/\rho_0\right)$$

$$= \nabla^2\left[C_s^2 + V_A^2\right]\left(\tilde{\rho}/\rho_0\right)$$

$$\boxed{\omega^2 = k_\perp^2\left(C_s^2 + V_A^2\right)}$$

$\Rightarrow$ "magnetosonic" or "compressional Alfven wave"

N.B. :

→ magnetosonic wave has $c^2 = C_S^2 + V_A^2$
⟺ combines acoustic, magnetic speeds
→ always faster (higher phase speed) than shear Alfven or acoustic mode.

i.e. $\underline{k} = \underline{k}_\perp$ magnetosonic wave is "fastest" MHD wave

→ recalling class discussion ? ⇒ how reconcile ?

- magnetosonic wave carried by field energy density → $B_0^2/8\pi\rho_0$

yet

- $V_{magn}^2 = V_A^2$, as in shear Alfven, which is carried by magnetic tension $B_0^2/4\pi\rho_0$.

Resolution : Freezing-in condition ⇒ $\underline{B}/\rho = \text{const}$, here
⇒ $\gamma_{eff} = 2$

i.e. freezing-in condition ⇒ field is stiff — indeed stiffer than gas, $\gamma = 5/3$ — acoustic medium i.e. $$C_S^2 = C_s^2 + C_B^2$$

$$= \frac{dP_{Th}}{d\rho} + \frac{dP_B}{d\rho}$$

$$= \gamma \frac{P_{Th_0}}{\rho_0} + 2\frac{P_B}{\rho_0}$$

i.e. for $\beta = P_{Th}/P_B = 1 \Rightarrow C_B^3 > C_s^2$.

So can summarize simple cases:

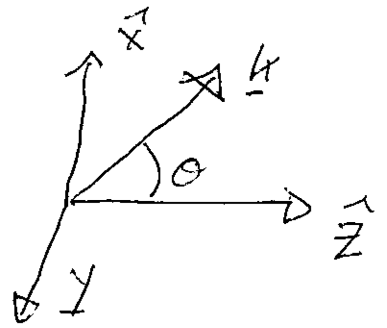
	$\nabla \cdot V = 0$	$\nabla \cdot \underline{V} \neq 0$
$\underline{k} = \underline{k}_{\parallel}$	$\omega^2 = k_{\parallel}^2 V_A^2$ shear Alfven	$\omega^2 = k_{\parallel}^2 C_s^2$ acoustic
$\underline{k} = \underline{k}_{\perp}$	X	$\omega^2 = k_{\perp}^2 (C_s^2 + V_A^2)$ magnetosonic wave.

Note that magnetosonic is 'fastest' of waves.

ii.) Full Crank — Read Kulsrud, Chapt. 5

Now, consider full crank for arbitrary $\underline{k}$.

geometry:



$$\begin{cases} \rho_0 = \rho_0 = \text{const} \\ \underline{B} = B_0 \hat{z} \end{cases}$$

have MHD equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$$

$$\rho \frac{d\underline{V}}{\partial t} = -\underline{\nabla} p + \frac{\underline{J} \times \underline{B}}{c}$$

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{V} \times \underline{B})$$

$$\frac{d}{dt}(p/\rho^\gamma) = 0 \quad \Rightarrow \quad \frac{1}{p}\frac{dp}{dt} - \frac{\gamma}{\rho}\frac{d\rho}{dt} = 0$$

and continuity $\Rightarrow$

$$\frac{1}{p}\frac{dp}{dt} = -\gamma\, \underline{\nabla} \cdot \underline{V}$$

Now, convenient to write $\underline{V}(\underline{x},t) = \frac{\partial \underline{\xi}}{\partial t}(\underline{x},t)$

$\underline{\xi}(\underline{x},t) \equiv$ displacement of fluid element, originally at $\underline{x}$ at t

$\Rightarrow$ with linearization $\tilde{\underline{V}} = \frac{\partial \underline{\xi}}{\partial t}$, $\rho = \rho_0 + \delta\rho$, etc.:

$$\delta\rho = -\rho_0 \underline{\nabla}\cdot\underline{\xi}$$

$$\delta P = -\gamma P_0 \underline{\nabla}\cdot\underline{\xi}$$

$$\delta \underline{B} = \nabla\times(\underline{\xi}\times\underline{B}_0)$$

$$\rho_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = -\underline{\nabla}\,\delta p + \left(\frac{\delta \underline{J}\times \underline{B}_0}{c}\right)$$

so can assemble the pieces, assuming $\underline{\xi} = \underline{\xi}_k e^{i(k\cdot x - \omega t)}$ and omitting subscript $\Rightarrow$

$\delta \underline{J} = \underline{\nabla}\times\delta\underline{B}$

$$\boxed{-\rho_0\omega^2\underline{\xi} = -\gamma P_0 \underline{k}(\underline{k}\cdot\underline{\xi}) - \frac{1}{4\pi}\left[\underline{k}\times(\underline{k}\times(\underline{\xi}\times\underline{B}_0))\right]\times\underline{B}_0}$$

from induction

- eigenmode equation for arbitrary displacement
- note as $\underline{\xi}$ is a 3 component vector above is $\underline{\underline{3}}$ linearly coupled equations. ω^2 is the eigenvalue. So ...

so — solution is det $|3\times3|$ $\Rightarrow$ cubic equation for ω^2. $\Rightarrow$ expect $\underline{\underline{3}}$ waves.

N.B.: Based on simple cases, what might there be?

$$\boxed{-\rho_0\omega^2\underline{\xi} = -\gamma P_0\underline{k}(\underline{k}\cdot\underline{\xi}) - \frac{1}{4\pi}\{\underline{k}\times[\underline{k}\times\underline{\xi}\times\underline{B_0}]\}\times\underline{B_0}}$$

$\rightarrow$ the 3 waves are, for the obvious profound reason, called the "fast", "slow" and "intermediate" waves...

— now, choose:
$$\begin{cases}\underline{k} = k(\sin\theta\,\hat{x} + \cos\theta\,\hat{z}) & \text{oblique in } xz \text{ plane}\\ \underline{\xi} = \xi\,\hat{y} & \text{i.e. } \underline{k}\cdot\underline{\xi}=0 \Rightarrow \underline{\nabla}\cdot\underline{V}=0\end{cases}$$

$\Rightarrow$ "intermediate wave" $\rightarrow$ clearly shear Alfven

now $\underline{k}\cdot\underline{\xi} = 0$

and crank $\Rightarrow$ $\left[\underline{k}\times[\underline{k}\times(\underline{\xi}\times\underline{B_0})]\right]\times\frac{\underline{B_0}}{4\pi}$

$$= \frac{(\underline{k}\cdot\underline{B_0})}{4\pi}[\underline{k}\times(\underline{\xi}\times\underline{B_0})]$$

$$= \frac{(\underline{k}\cdot\underline{B_0})}{4\pi}\,\underline{\xi}$$

so $\quad -\rho_0 \omega^2 \underline{\xi} = -\frac{(\underline{k}\cdot \underline{B}_0)^2}{4\pi} \underline{\xi}$

$\underline{\xi} = \xi_y \hat{y}$

$\Rightarrow \quad \omega^2 = k_{\parallel}^2 V_A^2 \quad$ with $\underline{\xi} = \xi_y \hat{y}$

shear Alfven $\rightarrow$ physical properties as before

$\therefore$ "intermediate wave" is shear Alfven

$\underline{\underline{\text{so}}}$ "fast wave" must connect to magnetosonic

$\therefore$ "slow wave" must connect to acoustic

$(c_s^2 < V_A^2)$

Let's see ..

— fast and slow waves:

again: $\quad \underline{k} = k(\sin\theta\, \hat{x} + \cos\theta\, \hat{z})$

$$\underline{\xi} = \xi_x \hat{x} + \xi_z \hat{z}$$

point here is that $\underline{k}\cdot\underline{\xi} \neq 0 \Rightarrow$ unlike intermediate these are compressional so now, crank ⇒

$$\frac{1}{4\pi}\left\{\underline{k}\times\left[\underline{k}\times(\underline{\xi}\times\underline{B}_0)\right]\right\}\times\underline{B}_0 = -\frac{k^2B_0^2}{4\pi}\,\xi_x\,\hat{x}$$

and

$$-\nabla p_1 = -\gamma p_0\,\underline{k}\,(\underline{k}\cdot\underline{\xi})$$

so

$$-\frac{\partial p_1}{\partial x} = -k^2\gamma p_0\left(\sin^2\theta\,\xi_x + \sin\theta\cos\theta\,\xi_z\right)$$

$$-\frac{\partial p_1}{\partial z} = -k^2\gamma p_0\left(\sin\theta\cos\theta\,\xi_x + \cos^2\theta\,\xi_z\right)$$

now, defining $\left.\begin{array}{l} c_s^2 = \gamma p_0/\rho_0 \\ v_A^2 = B_0^2/4\pi\rho_0 \end{array}\right\}$ as usual ⇒

$$\boxed{\begin{aligned} -\omega^2\xi_x &= -k^2\left(c_s^2\sin^2\theta + v_A^2\right)\xi_x - k^2c_s^2\sin\theta\cos\theta\,\xi_z \\ -\omega^2\xi_z &= -k^2c_s^2\sin\theta\cos\theta\,\xi_x - k^2c_s^2\cos^2\theta\,\xi_z \end{aligned}}$$

⇒ coupled equations for ξ_x, ξ_z

⇒ standard crank gives:

$$\begin{vmatrix} k^2v_A^2 + k^2c_s^2\sin^2\theta - \omega^2, & k^2c_s^2\sin\theta\cos\theta \\ k^2c_s^2\sin\theta\cos\theta, & k^2c_s^2\cos^2\theta - \omega^2 \end{vmatrix} = 0$$

and $$\boxed{\omega^4 - k^2 (C_s^2 + V_A^2)\omega^2 + k^4 C_s^2 V_A^2 \cos\theta = 0}$$

is "the dispersion relation".

Now can solve as:

$$\boxed{\frac{\omega^2}{k^2} = \frac{V_A^2 + C_s^2}{2} \pm \frac{1}{2}\left[(V_A^2 - C_s^2)^2 + 4 C_s^2 V_A^2 \sin^2\theta\right]^{1/2}}$$

→ upper root → "fast" wave
→ lower root → "slow" wave.

Now, check:

$\sin\theta = 0 \Rightarrow \underline{k} = k\hat{z}$

$$\frac{\omega^2}{k^2} = \frac{V_A^2 + C_s^2}{2} \pm \frac{(V_A^2 - C_s^2)}{2} \longrightarrow \begin{array}{l} V_A^2 \to \text{Alfven} \\ C_s^2 \to \text{acoustic} \end{array}$$

$\sin\theta = 1 \Rightarrow \underline{k} = k\hat{x}$

$$\frac{\omega^2}{k^2} = \frac{V_A^2 + C_s^2}{2} \pm \frac{1}{2}\left[(V_A^2)^2 + (C_s^2)^2 - 2V_A^2 C_s^2 + 4 C_s^2 V_A^2\right]^{1/2}$$

$$= \frac{V_A^2 + C_s^2}{2} \pm \frac{1}{2}\left[(V_A^2 + C_s^2)^2\right]^{1/2} = \begin{cases} 0 \\ V_A^2 + C_s^2 \end{cases}$$

Magnetosonic wave.

Note: can observe:

- for $\perp$ propagation, fast wave $\Leftrightarrow$ magnetosonic wave
  [slow = intermediate wave ; $\omega^2 = 0$]
- for $\parallel$ propagation, fast $\Leftrightarrow$ Alfven ($\beta < 1$)
  [fast = intermediate] slow $\Leftrightarrow$ acoustic ($\beta > 1$, vice versa)
- always have $V_{ph_{slow}} \leq V_{ph_{int}}^2 \leq V_{ph_{fast}}$

$\rightarrow$ have general result that polarizations of fast and slow modes are orthogonal

can show via:

$\mapsto$ matrix from eqns $\Leftrightarrow$ 2x2

$$-\rho \omega_s^2 \underline{\xi}_s = \underline{\underline{M}} \cdot \underline{\xi}_s \qquad (1)$$

$$-\rho \omega_f^2 \underline{\xi}_f = \underline{\underline{M}} \cdot \underline{\xi}_f \qquad (2)$$

$\underline{\xi}_f \cdot (1) - \underline{\xi}_s \cdot (2) \Rightarrow$

$$-\rho (\omega_s^2 - \omega_f^2)\, \underline{\xi}_s \cdot \underline{\xi}_f = \underline{\xi}_f \cdot \underline{\underline{M}} \cdot \underline{\xi}_s - \underline{\xi}_s \cdot \underline{\underline{M}} \cdot \underline{\xi}_f$$

but: recall from determinant

$$\underline{\underline{M}} = -\begin{bmatrix} k^2 V_A^2 + k^2 c_s^2 \sin^2\theta \ , & k^2 c_s^2 \sin\theta\cos\theta \\ k^2 c_s^2 \sin\theta\cos\theta \ , & k^2 c_s^2 \cos^2\theta \end{bmatrix}$$

and $\underline{\underline{M}}^T = \underline{\underline{M}}$ so $\underline{\underline{M}}$ self-adjoint!

$\Rightarrow$

$$\underline{\xi}_f \cdot \underline{\underline{M}} \cdot \underline{\xi}_s = \underline{\xi}_s \cdot \underline{\underline{M}} \cdot \underline{\xi}_f$$

$\hookrightarrow$ {important structural property in linear MHD

so $\underline{\xi}_f' \cdot \underline{\xi}_s' = 0$.

$\rightarrow$ to yet further elucidate the waves, can consider two limits $\beta \ll 1 \rightarrow c_s^2/V_A^2 \ll 1$
$\beta \gg 1 \rightarrow c_s/V_A^2 \gg 1$.

a) for $c_s^2 \gg V_A^2$,

l. order $\omega_f^2 = k^2 c_s^2$, $\omega_s = 0$

1st ord. $\frac{\omega_f}{k} \sim c_s + \frac{V_A^2 \sin^2\theta}{2c_s}$,

$\tilde{\xi} \parallel k$
(note $\xi_f \cdot \xi_s = 0$)

$\frac{\omega_s^2}{k^2} \approx V_A^2 \cos^2\theta$

$\tilde{\xi} \perp \underline{k}$
(otherwise $\tilde{\rho} \rightarrow$ higher ω)

b) for $c_s^2 \ll V_A^2$,

$$\frac{\omega_f^2}{k^2} \cong V_A^2 + c_s^2 \sin^2\theta$$

$\underline{\xi} \perp B_0$
(or no "springiness" to drive fast motion in parallel dir.)

$$\frac{\omega_s^2}{k^2} \simeq c_s^2 \cos^2\theta$$

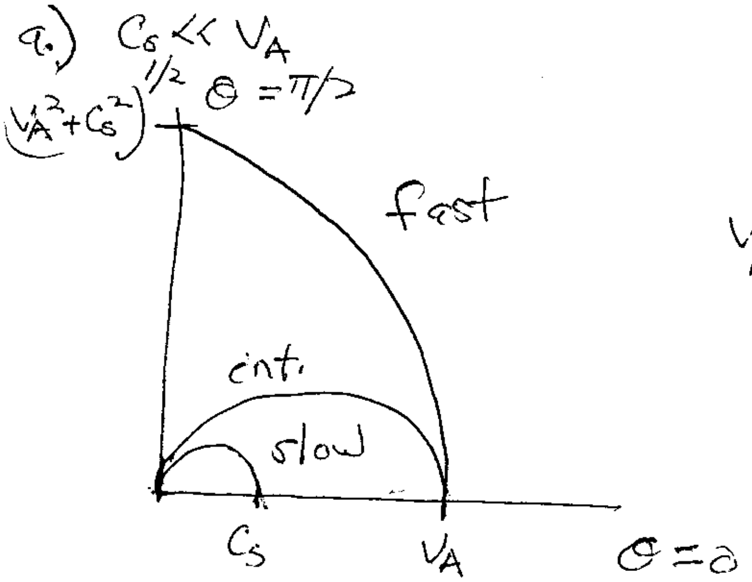
$\xi \parallel B_0$
(otherwise, if $\xi \perp B \Rightarrow$ get Alfven)

and again, $\underline{\xi}_s \cdot \underline{\xi}_f = 0$

$\Rightarrow$ now, can sum up this slow, intermediate, fast story in the Fredricks Diagram

consider $c_s \ll V_A$, $c_s \gg V_A$

a.) $c_s \ll V_A$



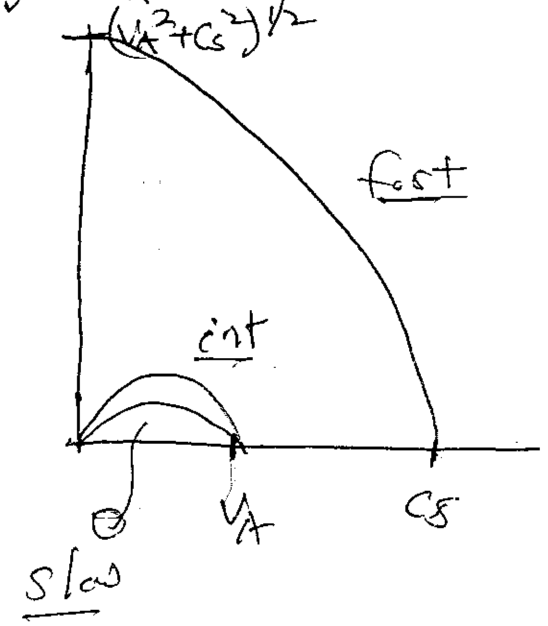
V_{phase} vs θ for:

fast $\Rightarrow$ magnetosonic at $\perp$
Alfven at $\parallel$

int $\Rightarrow$ Alfven at $\parallel$
nothing at $\perp$

slow $\Rightarrow$ acoustic (parallel) at $\parallel$
nothing at $\perp$.

b.) $V_A^2 \ll C_S^2$



again:

fast → magnetosonic at ⊥
Alfven at ∥

int. → Alfven at ∥
nothing at perp.

slow → Alfven at ∥
nothing at ⊥

→ now, observe the following:

→ 3 components $\underline{E}$
→ 2 component $\underline{B}$ $(\underline{\nabla}\cdot\underline{B}=0)$
→ P, ρ

⇒

∴ 7 fields

but 6 waves → 2 each $\left\{\begin{array}{l}\text{fast}\\ \text{intermediate}\\ \text{slow}\end{array}\right.$

$\omega^2 =$ ___

so, 1 missing mode! → entropy mode!

i.e $S = T \ln(p/\rho^{\gamma})$

and assumed $p_1/p_0 = \gamma \rho_1/\rho_0$

if relax ⇒ entropy wave $\left\{ \begin{array}{l} \delta\rho \neq 0, \text{ all else} = 0 \\ \omega = 0. \end{array} \right.$

relevant in shocks

→ some concluding philosophy ⇒ what is the moral of this story of the trip to the zoo of MHD waves?

- even for ≈ simple dynamical model like ideal MHD, even minimal anisotropy and produces great complexity!
- signal propagation $\left\{ \begin{array}{l} \text{parameter dependent} \\ \text{anisotropic} \\ \text{has definite polarization} \end{array} \right.$
- important to understand $\left\{ \begin{array}{l} \text{magnetic pressure} \\ \text{magnetic tension} \\ \text{thermal pressure} \end{array} \right.$

as origins of anisotropic restoring force in waves.

Aside

→ Reduced MHD → { Reduced Representation for strong ⓐ straight B_0 → eliminates fast mode

Note: ① full MHD : 3 $\underline{V}$ components
2 $\underline{B}$ " " ($\underline{\nabla} \cdot \underline{B} = 0$)
ρ, p

⇒ 7 components

② if $\underline{\nabla} \cdot \underline{V} = 0$ ⇒ 4 components
($\rho = \text{const}$, p from $\underline{\nabla} \cdot \underline{V} = 0$)

③ strongly magnetized system ⇒ Reduced MHD
⇒ scalar equations for ϕ, ψ (2 scalar fields)

Now:
- assume strong B_z (strong magnetization → gyrokinetics) → later

"strong" ⟷ $\rho v^2 \sim p \ll B_z^2/8\pi$

so motion strongly anisotropic, and small scales generated in $\perp$ direction only, as strong B_z inhibits line bending. (energy to perturb strong, high energy density field).

⇒ order : $B_z \sim \nabla_\perp \sim 1$

$$B_\perp \sim \partial_z \sim O(\epsilon)$$

Take $\rho \sim 1$, as $\underline{\nabla} \cdot \underline{V} = 0$ enforced by strong B_z.

$$V_\perp^2 \sim p \sim B_\perp^2 \qquad \text{(i.e. } \underset{\text{(springiness)}}{\text{equipartition}} \text{ of energy)}$$

$$\Rightarrow \quad V_\perp \sim \epsilon, \quad p \sim \epsilon^2, \quad \partial_t \sim \underline{V}_\perp \cdot \underline{\nabla}_\perp \sim \epsilon$$

and pressure balance ($\underline{\nabla} \cdot \underline{V} = 0$ and incompressibility)

$$\delta(B_z^2) \sim 2B_z \delta(B_z) \sim p \qquad \text{(eqbrm)}$$

i.e. $\omega \ll k(c_s^2 + V_A^2)^{1/2}$

$$\Rightarrow \delta B_z \sim \epsilon^2.$$

[idea is to order out the fast mode]

$\therefore$ to lowest order $\Rightarrow B_z = \text{const}$,

Now then: ($\underline{\nabla} \cdot \underline{B} = 0$)

$$\boxed{\begin{aligned} \underline{B} &= \hat{z} \times \underline{\nabla} \psi + B_z \hat{z} \\ &= \nabla A_\parallel \times \hat{z} + B_z \hat{z} \qquad \psi = -A_\parallel \end{aligned}}$$

B rep. by single scalar potential

parallel comp. of vector pot.

$$\underline{\nabla} \cdot \underline{B} = \partial_z \tilde{B}_z = \epsilon^3 \Rightarrow 0.$$

Similarly,

$$\partial_z p \sim O(\epsilon^3), \quad \Rightarrow \quad \cancel{V_z} \ll V_\perp$$

$$J_\perp B_\perp \sim \epsilon^3$$

neglect V_z.

Now, $\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi = -\frac{\underline{v}\times\underline{B}}{c}$

$$\Rightarrow \quad +\frac{1}{c}\frac{\partial \underline{A}}{\partial t} = \frac{\underline{v}\times\underline{B}}{c} - \underline{\nabla}\phi \qquad (*)$$

$$B_z = (\underline{\nabla}_\perp \times \underline{A}_\perp)\cdot\hat{z}$$

so $\partial_t A_\perp \sim \epsilon^3$ (a la $\partial_z P_z$)

$\therefore \quad \nabla_\perp \phi \cong \left(\frac{\underline{v}\times\underline{B}}{c}\right)_\perp$, in $(*)$

$$\Rightarrow \boxed{\underline{v}_\perp = \frac{c\hat{z}\times\underline{\nabla}\phi}{B_z}}$$

$\perp$ velocity

$\rightarrow$ motion $\perp$ is $\underline{E}\times\underline{B}$.

Now, taking parallel component of $(*)$.
(units!)

$$\Rightarrow \quad \frac{\partial}{\partial t}\psi + \underline{v}\cdot\underline{\nabla}\psi = B_z\partial_z\phi$$

so have (vector potential / flux) equation:

$$\boxed{\frac{\partial\psi}{\partial t} + \underline{v}\cdot\underline{\nabla}\psi = B_z\,\partial_z\phi}$$

$$= B_z\hat{z} + \hat{z}\times\nabla\psi$$

or, alternatively, $\boxed{\dfrac{\partial\psi}{\partial t} - \underline{B}\cdot\nabla\phi = 0}$,

Finally, for ϕ, write:

$\perp$ motion

cells of $E\times B$ drift,
('spin up' rate?)

$$\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\,\underline{V} = -\frac{\nabla p}{\rho_0} + \frac{\underline{J}\times\underline{B}}{c}$$

$(\nabla\times)\cdot\hat{z} \Rightarrow$ vorticity component ($\parallel \hat{z}$) evolution

$$\frac{\partial}{\partial t}\omega_z + \underline{V}_\perp\cdot\nabla\,\omega_z = -\cancel{\nabla\times\frac{\nabla p}{\rho_0}}^{\,0} + \hat{z}\cdot\nabla\times\left(\frac{\underline{J}\times\underline{B}}{c}\right)$$

$$= \underline{B}\cdot\nabla J_z - \cancel{\underline{J}\cdot\nabla B_z}$$

$\delta B_z \sim \epsilon^3$

$$\cong \underline{B}\cdot\nabla J_z$$

$\therefore$ $\boxed{\dfrac{\partial}{\partial t}\omega_z + \underline{V}\cdot\nabla\,\omega_z = \underline{B}\cdot\nabla J}$

but:

$$\omega_z = \hat{z}\cdot\nabla\times\underline{V} = \nabla^2\phi$$

$$J_z = \hat{z}\cdot(\nabla\times\underline{B})\frac{c}{\cancel{4\pi}} = \nabla^2\psi$$

so finally have:

$$\boxed{\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi = B_z\frac{\partial}{\partial z}\nabla^2\psi + \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi}$$

Finally, have reduced MHD equation:

$$\boxed{\begin{aligned}&\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\psi = B_z\,\partial_z\phi + \eta\nabla^2\psi \\ &\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi - \nu\nabla^2\nabla^2\phi \\ &\qquad = \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi + B_z\frac{\partial}{\partial z}\nabla^2\psi\end{aligned}}$$

- note have reduced MHD to 2 scalar evolution equations

- does this look familiar?

even stronger!

- for 2D MHD:

$$\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi = \underline{B}\cdot\underline{\nabla}\,\nabla^2\psi + \nu\nabla^2\nabla^2\phi$$

$$\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = \eta\nabla^2\psi$$

- ① Conservation Laws, etc. (HW)

$$\frac{d}{dt}E \underset{(\text{to } \eta,\nu)}{=} 0, \quad E = \int d^3x\left[\frac{(\nabla\phi)^2}{2} + \frac{(\nabla\psi)^2}{2}\right]$$

② $H = A\cdot B \cong \underset{\text{const.}}{B_z}\,\psi$

$\Rightarrow \quad H = \int d^3x\, B_z\psi, \quad \frac{dH}{dt} = 0$, to $O(\eta)$

Ohm's Law (flux advection) is simple statement of helicity conservation.

? $\frac{\partial}{\partial t}\psi + \nabla\cdot\Gamma_\psi = \eta\nabla^2\psi$ form Γ_ψ s/t $\begin{cases} H \text{ conserved} \\ E_M \text{ dissipated}\end{cases}$

③ $K = \int d^3x\,\underline{V}\cdot\underline{B} = \int d^3x\,(\nabla\phi\cdot\nabla\psi)$

also conserved, to dissipation.

B.) 'Least Action' and the Energy Principle in MHD

→ Introduction

- we now arrive at the MHD Energy Principle, which is a highlight of MHD, plasma physics and classical physics, in general.

- Energy Principle → stability

i.e. till now
- 218B – waves, etc.
- 218A – trivial instabilities (i.e. 2-stream, bump-on-tail, J-driven ion-acoustic)

realistic plasmas {lab or astro} → free energy (∇P, ∇J etc.) + complex geometry, b.c.'s, etc.

→ instabilities with complex dynamics ...

i.e.
- Rayleigh-Benard → ∇S
- Interchanges → K, ∇P (includes Rayleigh-Taylor)
- kinks, tearing → ∇J, $q(r)$

→
- relaxation, turbulence, shocks ...
- limits on performance (lab)
- restrictions on morphology (lab and astro)

- brute force, frontal assault on instabilities often leads to heavy casualties ....

$\therefore$

- need a simple criterion, i.e. a necessary/sufficient criterion to identify and characterize instabilities

$\Rightarrow$ Energy Principle !

- Energy Principle is very much in spirit of R-R variational principle $\leftrightarrow$ no surprise as both based on self-adjointness of linear operator

- Proceed via:

- sketch of Principle of Least Action for Ideal MHD
  $\Rightarrow$ Lagrangian formulation (Kulsrud 4.7)

N.B. This underlies formulation in terms of displacement...

- MHD eigenmode equation (generalizes simple wave studies so far), second order W

$\Rightarrow$

- energy principle. (Kulsrud 7.1, 7.2)
(Kadomtsev Article)

— applications (various)

i) Principle of Least Action for MHD

— For ideal MHD, can immediately write

$$L = \int d^3x \left[\frac{\rho v^2}{2} \right] - W \qquad \text{(Lagrangian)}$$

$$S = \int dt\, L$$
↳ action

$$W = \int d^3x \left(\frac{P}{\gamma - 1} + \frac{B^2}{8\pi} + \rho\phi \right)$$

so $$\mathcal{L} = \frac{\rho v^2}{2} - \left(\frac{P}{\gamma-1} + \frac{B^2}{8\pi} + \rho\phi \right)$$

and can derive MHD equations by $\delta L = 0$
i.e. Principle of Least Action

— key point: how parametrize trajectory variations ??

i.e. for string: (easy)

$$L = \int dt \int_0^L dx \left[\frac{1}{2}\left(\frac{\partial y}{\partial t}\right)^2 - T\left[\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{1/2} - 1\right]\right]$$

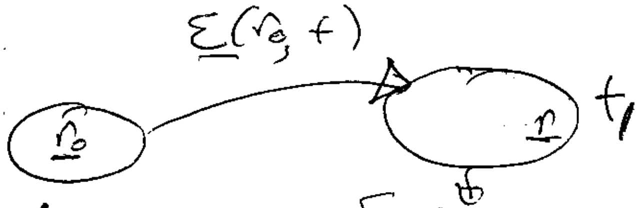
$$\delta L = \delta L/\delta y \Rightarrow \frac{\partial L}{\partial y_t}\delta y_t + \frac{\partial L}{\partial y_x}\delta y_x \quad \text{etc...}$$

$\therefore$ analogy with string suggests displacement is!

→ natural way to formulate Least Action for ideal MHD

→ natural link of MHD dynamics to particle dynamics

i.e.



[at t_1, original blob has $\underline{r} = \underline{r}_0 + \underline{\xi}(\underline{r}_0, t)$]

- how relate $\underline{\xi}(\underline{r}_0, t)$ to Eulerian velocity?

i.e. during δt, fluid element moves

from $\quad\quad\quad\quad\quad\quad\quad$ to

$$r = \underline{r}_0 + \underline{\xi}(\underline{r}_0, t) \longrightarrow \underline{r}_0 + \underline{\xi}(\underline{r}_0, t) + \left(\partial \underline{\xi}/\partial t\right)\delta t$$

$$\therefore \quad \underline{V}(r_0 + \underline{\xi}(r_0, t), t) = \frac{\partial \underline{\xi}}{\partial t}(\underline{r}_0, t)$$

→ 3 components of $\underline{\xi}$ satisfy 3 nonlinear odes with $\underline{\xi}(\underline{r}_0, t_0) = 0$ as i.c.

→ theory of ode's assures solution exists.

Now, as in wave theory, can write all changes in MHD quantities in terms of displacements, i.e.

$$\delta\rho = -\underline{\nabla}\cdot\left[\rho(\underline{r},t)\,\delta\underline{\xi}(\underline{r},t)\right]$$

$$\delta p = -\gamma p(\underline{r},t)\underline{\nabla}\cdot\delta\underline{\xi}(\underline{r},t) - \delta\underline{\xi}(\underline{r},t)\cdot\underline{\nabla}p(\underline{r},t)$$

$$\delta\underline{B} = \underline{\nabla}\times\left(\delta\underline{\xi}(\underline{r},t)\times\underline{B}(\underline{r},t)\right)$$

and

$$\delta V(\underline{r},t) = \underline{V}(\underline{r},t)\cdot\nabla\,\delta\underline{\xi}(\underline{r},t) - \delta\underline{\xi}(\underline{r},t)\cdot\underline{\nabla}\,\underline{V}(\underline{r},t) + \partial\,\delta\underline{\xi}(\underline{r},t)/\partial t$$

so now, can consider δS

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x\, \delta\mathcal{L}$$

$$= \int_{t_1}^{t_2} dt \int d^3x \left(\delta\rho\,\frac{\underline{V}^2}{2} + \rho\underline{V}\cdot\delta\underline{V} - \frac{\delta p}{\gamma-1} - \frac{\underline{B}\cdot\delta\underline{B}}{4\pi} - \delta\rho\,\phi\right)$$

plugging in δ quantities $\Rightarrow$

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x \left\{ \overbrace{\nabla\cdot(-\rho\,\delta\underline{\xi})\frac{V^2}{2} + \rho\underline{V}\cdot\left(\underline{V}\cdot\underline{\nabla}\delta\underline{\xi} - \delta\underline{\xi}\cdot\underline{\nabla}\underline{V} + \frac{\partial \delta\underline{\xi}}{\partial t}\right)}^{\delta KE} \right\} + \int_{t_1}^{t_2} dt\, d^3x \overbrace{\left(\frac{\gamma p\,\underline{\nabla}\cdot\delta\underline{\xi} + \delta\underline{\xi}\cdot\underline{\nabla}p}{\gamma-1}\right)}^{\delta\,Th\,E}$$

$$- \int_{t_1}^{t_2} dt \int d^3x\, \overbrace{\frac{\underline{B}\cdot\underline{\nabla}\times(\delta\underline{\xi}\times\underline{B})}{4\pi}}^{\delta E_B} + \int_{t_1}^{t_2} dt \int d^3x\, \nabla\cdot(\rho\,\delta\xi)\,\phi$$

Now $\delta\xi\Big|_{t_1,t_2} = 0$, $\delta\xi\Big|_{bndry} = 0$

so ∂ by parts a lot $\Rightarrow$ (with b.c.'s)

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x \left\{ \delta\underline{\xi}\cdot\left[\rho\nabla\frac{\underline{V}^2}{2} - \underline{\nabla}\cdot(\rho\underline{V}\,\underline{V}) - \rho\underline{\nabla}\frac{\underline{V}^2}{2} - \frac{\partial}{\partial t}(\rho\underline{V})\right] - \frac{\delta\underline{\xi}\cdot\gamma\underline{\nabla}p + \delta\underline{\xi}\cdot\nabla p}{(\gamma-1)} - \delta\underline{\xi}\cdot\rho\underline{\nabla}\phi + \delta\underline{\xi}\cdot\frac{(\underline{\nabla}\times\underline{B})\times\underline{B}}{4\pi} \right\}$$

∇p $\quad$ $\rho\nabla\phi$ $\quad$ $\underline{J}\times\underline{B}$

so

$$\delta S = -\int_{t_1}^{t_2} dt \int d^3x \; \delta \underline{\xi} \cdot \left[\frac{\partial}{\partial t}(\rho \underline{V}) + \nabla \cdot (\rho \underline{V}\,\underline{V}) + \underline{\nabla} p - \underline{J} \times \underline{B} + \rho \underline{\nabla} \phi \right]$$

so

$$\delta S = 0 \quad \text{and} \quad \delta \underline{\xi} \neq 0 \Rightarrow$$

$$\frac{\partial}{\partial t}(\rho \underline{V}) + \underline{\nabla} \cdot (\rho \underline{V}\,\underline{V}) = -\underline{\nabla} p + \underline{J} \times \underline{B} - \rho \underline{\nabla} \phi$$

and $\dfrac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\rho \underline{V}) \Rightarrow$

$$\boxed{\rho \left[\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{V} \right] = -\underline{\nabla} p + \underline{J} \times \underline{B} - \rho \underline{\nabla} \phi}$$

$\Rightarrow$ equation of motion of ideal MHD emerges as "Lagrange's Equation".

Note: for case of $\underline{V} = 0$ $\rightarrow$ equilibrium solution then $\delta S = 0$ gives:

$$\underline{\nabla} p = \underline{J} \times \underline{B} - \rho \underline{\nabla} \phi$$

Moral of this story:

→ can derive MHD equations from Principle of Least Action

→ displacement is a useful way to formulate ideal MHD dynamics

Now this brings us to:

ii.) <u>Energy Principle</u> - Simple Form

Consider inhomogeneous static equilibrium/initial state with:

$\uparrow \rho$ const g. $(c \to 1)$

eqbrm conditions

$$\left\{\begin{array}{l} \underline{\nabla} P_0 = \underline{J}_0 \times \underline{B}_0 - \underline{\rho}\,\underline{g} \\ \underline{\nabla} \times \underline{B}_0 = 4\pi \underline{J}_0 \\ \underline{\nabla} \cdot \underline{B}_0 = 0 \end{array}\right.$$

all $\left\{\begin{array}{l} P_0 = P_0(r_0) \\ \text{time independent} \\ \text{but inhomogeneous!} \end{array}\right.$

and no flow or $\left\{\begin{array}{l}\text{self-} \\ \text{gravity}\end{array}\right.$ ......

Further assume → rigid wall bounds system (!?)

$$\rightarrow \underline{V} \cdot \hat{n} \Big|_{wall} = 0$$

$$\underline{B} \cdot \hat{n} \Big|_{wall} = 0$$

and now ... $\Rightarrow$ perturb system from eqbrm by $\underline{\xi}$

$\Rightarrow$ so, at $t=0$:

$$\underline{\xi}(r) = \xi_0(\underline{r})$$

$$\frac{\partial \xi(r)}{\partial t} = \hat{\xi}_0(r)$$

$\Rightarrow$ keep only linear terms in $\underline{\xi}$ $\Rightarrow$

$$\underline{r} = \underline{r_0} + \xi(\underline{r_0}, t)$$

and $\underline{r_0} \to \underline{r}$ in argument of perturbed quantities.

so

$\Rightarrow$

$$\rho(t, r) = \rho_0 - \nabla\cdot(\rho_0 \underline{\xi})$$

$$P(t, \underline{r}) = P_0 - \gamma P_0 \underline{\nabla}\cdot\underline{\xi} - \underline{\xi}\cdot\underline{\nabla} P_0$$

$$\underline{B}(t, \underline{r}) = \underline{B_0} + \underline{\nabla}\times(\underline{\xi}\times\underline{B_0})$$

$$4\pi J(\underline{r}, t) = \underline{j_0} + \underline{\nabla}\times\left[\underline{\nabla}\times(\underline{\xi}\times\underline{B_0})\right]$$

so putting it into equation of motion (linearized) $\Rightarrow$

$$\boxed{\rho_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = \underline{F}(\underline{\xi})}$$

where:

$$F(\xi) = \frac{1}{4\pi}\Big[\underbrace{\underline{\nabla}\times[\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)]\Big)\times B_0}_{\delta \underline{J}\times B_0}$$

$$+ \underbrace{\underline{J}_0 \times [\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)]}_{\underline{J}\times\delta\underline{B}} \quad \underbrace{-\underline{g}\, \underline{\nabla}\cdot(\rho_0 \underline{\xi})}_{-g\,\delta\rho}$$

$$\underbrace{+ \underline{\nabla}\left[\underline{\xi}\cdot\underline{\nabla}P_0 + \gamma P_0 (\underline{\nabla}\cdot\underline{\xi})\right]}_{-\underline{\nabla}\tilde{P}}$$

with b.c. $\begin{cases} \underline{\xi}\cdot\vec{n} = 0 & \text{on surface} \\ \underline{B}\cdot\vec{n} = 0 & \text{on surface} \end{cases}$

Key Point:

$\rightarrow$ $\boxed{\underline{F}(\underline{\xi}) \text{ is self-adjoint}}$!!

i.e.

$$\int d^3x\ \underline{\eta}\cdot\underline{F}(\xi) = \int d^3x\ \underline{\xi}\cdot F(\underline{\eta})$$

→ to prove: see Kulsrud, Pblm. 6 (coming on Pblm Set III)

or consider the following (an indirect proof)...

legerdemain involved...

→ can write total energy, to second order (in displacement) as:

i.e. $E = \int d^3x \, \frac{\rho_0(r)}{2} \left(\frac{\partial \xi}{\partial t}\right)^2 + W(\xi, \xi)$

2nd order bit of:

$\int \left(\frac{p}{\gamma - 1} + \frac{B^2}{8\pi} + \rho\phi \right) d^3x$

Now:

→ $W = W_0 + W_1(\xi) + W_2(\xi, \xi)$

first order — second order

→ total energy is conserved, for any ξ with initial conditions ξ_0, $\dot{\xi}_0$, provided $\xi \cdot \hat{n} = \dot{\xi} \cdot \hat{n} = 0$ (b.c.)

Now, $dE/dt = 0 \Rightarrow$

$$\frac{dE}{dt} = \int_V d^3x \, \rho_0 \left\{ \frac{\partial \underline{\xi}}{\partial t} \cdot \frac{\partial^2 \underline{\xi}}{\partial t^2} \right\} + W_1 \left(\frac{\partial \underline{\xi}}{\partial t} \right)$$

$$+ W_2 \left(\frac{\partial \underline{\xi}}{\partial t}, \underline{\xi} \right) + W_2 \left(\underline{\xi}, \frac{\partial \underline{\xi}}{\partial t} \right) = 0$$

and $\rho_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = \underline{F}(\underline{\xi}) \Rightarrow$

$$\frac{dE}{dt} = \int d^3x \left[\frac{\partial \underline{\xi}}{\partial t} \cdot \underline{F}(\underline{\xi}) \right] + W_1 \left(\frac{\partial \underline{\xi}}{\partial t} \right)$$

$$+ W_2 \left(\frac{\partial \underline{\xi}}{\partial t}, \underline{\xi} \right) + W_2 \left(\underline{\xi}, \frac{\partial \underline{\xi}}{\partial t} \right)$$

but since $dE/dt = 0$ is always true, it is true at $t=0$, a particular time

setting $\dot{\underline{\xi}}_0 \equiv \underline{\eta} \Rightarrow$

$\hookrightarrow$ a particular displ. ...

$$\int d^3x \, \underline{\eta} \cdot \underline{F}(\underline{\xi}) + W_1(\underline{\eta}) + W_2(\underline{\eta}, \underline{\xi})$$

$$+ W_2(\underline{\xi}, \underline{\eta}) = 0$$

now, $W_1(\underline{\eta}) = 0$ <u>so</u> (no velocity dependence on i.c.)

$$\int d^3x \, \underline{\eta} \cdot F(\underline{\xi}) + [W_2(\underline{\eta}, \underline{\xi}) + W_2(\underline{\xi}, \underline{\eta})] = 0$$

or more clearly $\Rightarrow$

$$\int d^3x \, \underline{\eta} \cdot F(\underline{\xi}) = -[W_2(\underline{\eta}, \underline{\xi}) + W_2(\underline{\xi}, \underline{\eta})]$$

so RHS symmetric under $\underline{\eta} \leftrightarrow \underline{\xi}$ interchange

<u>so</u> so is LHS ! i.e.

$$\int d^3x \, \underline{\eta} \cdot F(\underline{\xi}) = \int d^3x \, \underline{\xi} \cdot F(\underline{\eta})$$

and have proved self-adjointness !

→ finally, useful to note that if now $\underline{\eta} = \underline{\xi}$

$$\boxed{W_2(\underline{\xi}, \underline{\xi}) = -\frac{1}{2} \int d^3x \, [\underline{\xi} \cdot F(\underline{\xi})]}$$

— a handy expression for W_2 in terms F !

so now, have shown that:

→ $\underline{F}(\underline{\xi})$ self-adjoint

→ $W_2(\underline{\xi})$, the potential energy of displacement $\underline{\xi}$, can be expressed as:

$$W_2(\underline{\xi}) = -\frac{1}{2}\int d^3x \left[\underline{\xi}\cdot\underline{F}(\underline{\xi})\right]$$

From these, we show several important results:

- reality of ω^2 and "exchange of stabilities"
  ↔ clue to structure of instability in ideal MHD

- orthagonality of eigenfunctions

- Variational structure

i) reality of ω^2, "exchange of stabilities"

$$\underline{\xi} = \tilde{\underline{\xi}}(x)\, e^{-i\omega t}$$

$$-\rho_0\omega^2\,\underline{\xi} = \underline{F}(\underline{\xi}) \qquad (1)$$

$$-\rho_0\omega^{2*}\,\underline{\xi}^* = \underline{F}(\underline{\xi}^*) \qquad (2)$$

n.b. F is explicitly real

$\underline{\xi}^* \cdot (1) - \underline{\xi} \cdot (2) \Rightarrow$

$$-\rho_0(\omega^2-\omega^{2*})\,\underline{\xi}^*\cdot\underline{\xi} = \underline{\xi}^*\cdot\underline{F}(\underline{\xi}) - \underline{\xi}\cdot\underline{F}(\underline{\xi}^*)$$

and integrating $\Rightarrow$

$$-\rho_0(\omega^2-\omega^{2*})\int d^3x\,(\underline{\xi}^*\cdot\underline{\xi}) = \int d^3x\left[\underline{\xi}^*\cdot\underline{F}(\underline{\xi}) - \underline{\xi}\cdot\underline{F}(\underline{\xi}^*)\right]$$

$= 0$, by self-adjoint property

as $\underline{\xi}^*\cdot\underline{\xi}$ real $\Rightarrow$ $(\omega^2)^* = \omega^2$

$\Rightarrow$ ω^2 is real

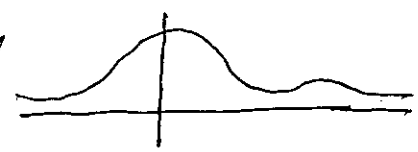
$\therefore$ $\omega^2 > 0$ $\rightarrow$ stability

$\omega^2 < 0$ $\rightarrow$ instability, but purely growing
no oscillation

Contrast to instabilities with which you should be familiar:

$\rightarrow$ bump-on-tail $\qquad \omega = \omega_k^0 + i\gamma_k$

$\gamma \sim \partial f_0/\partial v$



Wave + inverse dissipation

$\downarrow$

carrier

→ two stream $\quad \epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - k v_0)^2}$

~ coupling of $\begin{cases} \text{positive energy wave in plasma} \\ \text{negative energy wave in beam} \end{cases}$

- "reactive" counter-part of bump on tail ⇔ can have ω^2 real

⇒ beam + dissipation ⇒ negative energy wave ⊕ dissipation ⇒ growth

$\omega = \omega_r + i\gamma$

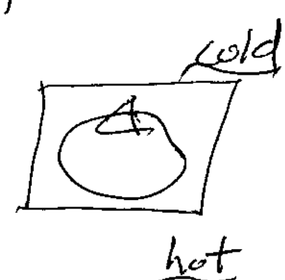
⇒ current-driven ion-acoustic

$\omega = \omega_r + i\gamma \qquad \gamma = (\,)\frac{\partial f_e^{(0)}}{\partial v} - (\,)\frac{\partial f_i^{(0)}}{\partial v}$

wave + competition of dissipation and dissipation

⇒ ideal Rayleigh-Benard Convection

$\omega^2 = -\frac{k_H^2}{k_H^2 + k_V^2}\, g \frac{\partial S'}{\partial z} \qquad \nabla_z S < 0$



$\downarrow g > 0$

∴ of these, ideal MHD instabilities similar in structure to convection and ω^2 real cases of 2-stream, and <u>different</u> in structure from the others

Extra Credit: For the few, the proud, the insane..

a.) Can one develop an energy principle approach to the two-stream instability, using cold fluid equations? Beware V_0 and self-adjointness..

b.) If yes, extend your result to include finite pressure (i.e. warm plasma effects) in the target plasma. If not, explain in detail why not...

⇒ In ideal MHD, instability defines structure of eigenfunction, i.e. $\tilde{\underline{\xi}} = \tilde{\underline{\xi}}(\underline{r}, \gamma)$.

N.B. In ideal MHD, only scale in problem is system size ⇔ boundaries. Contrast Sweet-Parker reconnection ($\Delta/L < 1$), a case of resistive MHD.

proceeding ⇒

Since ω^2 real, ω^2 must pass thru $\omega^2 = 0$ as the system evolves from stable to unstable.

- this evolution is called "exchange of stabilities"

- ⇒ marginal displacement solves $\underline{F}(\underline{\xi}) = 0$

N.B. $\Rightarrow$ Solution of $\underline{F}(\underline{\xi}) = 0$ gives linear stability boundary, in parameter space

ii.) orthogonality

consider two solutions to $\quad -\rho_0 \omega^2 \underline{\xi} = \underline{F}(\underline{\xi})$,

$$-\rho_0 \omega_1^2 \underline{\xi}_1 = \underline{F}(\underline{\xi}_1) \qquad \underline{\times}\ \underline{\xi}_2$$

$$-\rho_0 \omega_2^2 \underline{\xi}_2 = F(\underline{\xi}_2) \qquad \underline{\times}\ \underline{\xi}_1$$

$$-(\omega_1^2 - \omega_2^2) \int d^3x \rho_0 \underline{\xi}_1 \cdot \underline{\xi}_2 = \int d^3x \left[\underline{\xi}_2 \cdot \underline{F}(\underline{\xi}_1) - \underline{\xi}_1 \cdot \underline{F}(\underline{\xi}_2) \right]$$

$$= 0 \text{ , by self-adjointness}$$

$$\omega_1^2 \neq \omega_2^2 \Rightarrow \int d^3x\ \rho_0\ \underline{\xi}_1 \cdot \underline{\xi}_2 = 0$$

$\Rightarrow$ orthonormality, with weighting function ρ_0.

The point of all this is that now we can set up a variational quadratic form, ala' beloved Sturm-Liouville theory

$$-\rho\omega^2 \underline{\xi} = \underline{F}(\underline{\xi})$$

and $\otimes\ \dfrac{\underline{\xi}}{2}\cdot \Rightarrow$

$$\boxed{\omega^2 = \frac{-\int d^3x\ \underline{\xi}\cdot \underline{F}(\underline{\xi})/2}{\int \rho_0\, \underline{\xi}^2/2}}$$

$$= W_2(\underline{\xi}) \Big/ \int \rho\, \xi^2/2 .$$

$\Rightarrow$ with $K(\underline{\xi}) \equiv \int d^3x\ \rho\, \xi^2/2$, have

$$\boxed{\omega^2 = W_2(\underline{\xi}) / K(\underline{\xi})} \rightarrow \left\{ \begin{array}{l} \text{variational, quadratic} \\ \text{form} \end{array} \right.$$

and we know that, since all requirements satisfied, that

$\rightarrow$ any trial $\underline{\xi}$ plugged into $W_2(\underline{\xi})/K(\underline{\xi})$ — yields $\omega^2(\underline{\xi}) > \omega_T^2$

$\hookrightarrow$ the true eigenvalue.

i.e. variational result is always upper bound.

→ so, we know that

- if can find a trial $\underline{\xi}$ such that

$$W_2(\underline{\xi}) < 0$$

- then, configuration is surely unstable

∴ this yields the desired necessary and sufficient condition for instability namely that it be possible to find a $\underline{\xi}$ such that

$$W_2(\underline{\xi}) < 0.$$

hereafter, we write $W_2(\underline{\xi}) = \delta W(\underline{\xi})$,

so the MHD Energy Principle is just:

$$\boxed{\begin{array}{l}\text{instability iff } \exists \text{ well behaved } \underline{\xi} \text{ s/t} \\ \quad \delta W(\underline{\xi}) < 0\end{array}}$$

N.B.

- in physical terms, E.P. $\Rightarrow$ instability if can find a displacement which lowers the energy. Note linear instability $\Longleftrightarrow$ δW to $\mathcal{O}(\xi^2)$ considered

- know $\delta W(\xi) = -\frac{1}{2}\int d^3x\, \xi \cdot F(\xi)$

so, now must manipulate δW into physically useful form, i.e. recall

$$F(\xi) = \frac{1}{4\pi}\underbrace{\left\{ \nabla \times [\nabla \times (\xi \times B_0)] \right\} \times B_0}_{\delta J \times B_0 \quad -①}$$

$$+ \underbrace{J_0 \times [\nabla \times (\xi \times B_0)]}_{J_0 \times \delta B \quad -②}$$

$$+ \underbrace{\nabla [P_0 \nabla \cdot \xi + \xi \cdot \nabla P_0]}_{\nabla \delta p \quad -③} + \underbrace{\nabla \cdot (\rho_0 \xi) \nabla \phi}_{\delta\rho \nabla \phi \quad -④}$$

$$= F_① + F_② + F_③ + F_④$$

remember here, all B_0, P_0, ρ_0 etc. inhomogeneous, and $\xi \cdot \hat{n}$ and $B \cdot \hat{n}$ on boundary.

- remains to manipulate $-\int [\underline{\xi} \cdot F(\underline{\xi})/2] \, d^3x$ into "illuminating" form

- key is sign of δW, so seek to extract quadratic terms, as unambiguous.

$\Rightarrow$ let the crank begin!

① $$\delta W_{①} = -\frac{1}{2}\int \underline{\xi} \cdot \underline{F}_{①}(\underline{\xi}) \, d^3x$$

$$= -\frac{1}{2}\int d^3x \, \frac{\underline{\xi}}{4\pi} \cdot \left\{ \left(\underline{\nabla} \times [\nabla \times (\underline{\xi} \times \underline{B}_0)] \right) \times \underline{B}_0 \right\}$$

$$= \frac{1}{8\pi}\int d^3x \, \left(\underline{\nabla} \times [\underline{\nabla} \times (\underline{\xi} \times \underline{B}_0)] \right) \cdot \underline{\xi} \times \underline{B}_0$$

$$= \frac{1}{8\pi}\int d^3x \, \nabla \cdot [\nabla \times (\underline{\xi} \times \underline{B}_0) \times (\underline{\xi} \times \underline{B}_0)] + \frac{1}{8\pi}\int d^3x \, \left(\underline{\nabla} \times (\underline{\xi} \times \underline{B}_0) \right) \cdot \left(\underline{\nabla} \times (\underline{\xi} \times \underline{B}_0) \right)$$

if $\underline{Q} \equiv \nabla \times (\underline{\xi} \times B_0) = \delta \underline{B}$, from induction

$$\delta W_{①} = \int d^3x \, \frac{\underline{Q}^2}{8\pi} + \frac{1}{8\pi}\int d\underline{S} \cdot (\nabla \times (\underline{\xi} \times B_0)) \times (\underline{\xi} \times \underline{B}_0)$$

$$\Rightarrow \quad \delta W_{1}\Big|_{\text{surface}} = \frac{-1}{8\pi}\int ds \left[\cancel{\hat{n}}\cdot\underline{B}_0 \;\; \underline{\xi}\cdot\underline{Q} - \cancel{(\hat{n}\cdot\underline{\xi})}\, \underline{B}_0\cdot\underline{Q} \right]$$

$$\delta W_{1} = \int d^3x \; \frac{Q^2}{8\pi}$$

$$\delta W_{2} = -\frac{1}{2}\int d^3x \;\; \underline{\xi}\cdot\underline{J}_0 \times \left[\nabla\times(\underline{\xi}\times\underline{B}_0)\right]$$

$$= -\frac{1}{2}\int d^3x \;\; \underline{\xi}\cdot(\underline{J}_0\times\underline{Q})$$

$$= +\frac{1}{2}\int d^3x \;\; \underline{J}_0\cdot(\underline{\xi}\times\underline{Q})$$

$$\delta W_{3} = -\frac{1}{2}\int d^3x \;\; \underline{\xi}\cdot\underline{\nabla}\left[p_0\,\underline{\nabla}\cdot\underline{\xi} + \underline{\xi}\cdot\underline{\nabla}p_0\right]$$

ibp $\underline{\xi}\cdot\hat{n} = 0$ on boundary

$$\Rightarrow \delta W_{3} = \int \frac{d^3x}{2}\left[p_0(\underline{\nabla}\cdot\underline{\xi})^2 + (\underline{\nabla}\cdot\underline{\xi})\,\underline{\xi}\cdot\underline{\nabla}p_0\right]$$

and last but not least...

$$\delta W_{④} = -\int \frac{d^3x}{2} \; \underline{\xi} \cdot \nabla \cdot (p_0 \underline{\xi}) \underline{\nabla} \phi$$

$$= -\frac{1}{2} \int d^3x \; (\underline{\xi} \cdot \underline{\nabla} \phi) \; \nabla \cdot (p_0 \underline{\xi})$$

so, putting the whole mess together

$$\boxed{\begin{aligned} \delta W = \frac{1}{2} \int d^3x \Big\{ & \overset{①}{\frac{\underline{Q}^2}{4\pi}} + \overset{②}{\underline{J}_0(\underline{x}) \cdot (\underline{\xi} \times \underline{Q})} \\ & + \overset{③}{\gamma p_0(\underline{x}) (\underline{\nabla} \cdot \underline{\xi})^2} + \overset{④}{(\underline{\xi} \cdot \underline{\nabla} p_0(\underline{x})) \underline{\nabla} \cdot \underline{\xi}} - \overset{⑤}{(\underline{\xi} \cdot \underline{\nabla} \phi) \nabla \cdot (p_0 \underline{\xi})} \Big\} \\ \underline{Q} = \nabla \times (\underline{\xi} \times \underline{B}_0) & \end{aligned}}$$

note: general characteristics

- ① $\rightarrow$ > 0 $\rightarrow$ field line bending
  ③ $\rightarrow$ > 0 $\rightarrow$ compression
  $\Big\}\rightarrow$ always stabilizing $\delta W > 0$

- free energy sources:

  $\underline{J}_0(x)$ in ② $\rightsquigarrow$ current profile

  $\underline{\nabla} p_0(x)$ in ④ $\rightsquigarrow$ pressure gradient

$\Rightarrow$ can make $\delta W < 0$, for certain profiles and $\underline{\xi}$ $\Rightarrow$ free energy sources for instability.

Note:

→ δW is imprecise

→ δW does not reveal much about growth rates

but

→ very useful for simple quick assessment of stability

→ can elucidate
- complex problem
- problem in which intuition re: equilibrium not precise.

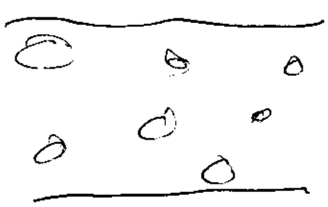
∴ furthur developments in theory remain, but better to consider some examples
⇒

iii) Convection and Interchange Instabilities
→ A Simple Application of the Energy Principle

consider 4 related examples:

a.) Convection and the Schwarzschild Criterion
b.) Rayleigh-Taylor Instability
c.) Interchange Instability
d.) Interchange Without Gravity

a.) Schwarzschild Criterion and Convection
i.e. stellar atmosphere



$$\left(\rho g = \frac{dP}{dz} \right)$$

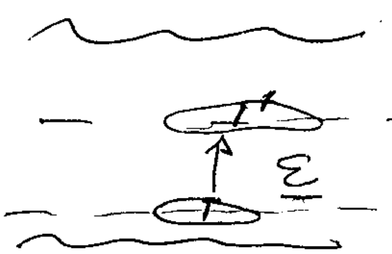
$z \uparrow \qquad \frac{d\rho}{dz} < 0, \ \frac{dP}{dz} < 0$

and

$$P \rho^{-\gamma} = \text{const.}$$

(basic means of heat transport)

For basic idea of convection, consider a virtual displacement of a slug/blob of gas upward
⇒ physical argument



if $\rho_1' < \rho_2 \Rightarrow$ blob sinks
system stable

$\rho_1' > \rho_2 \Rightarrow$ blob rises,
system unstable

for infinitesimal displacement, $\varepsilon \sim \Delta z \Rightarrow$

$$\rho_2 = \rho_1 + \frac{d\rho_1}{dz}\Delta z$$

For ρ_1', $\rightarrow$ system is isentropic $\Rightarrow$
$P\rho^{-\gamma} = \text{const}$ applies

$\Rightarrow$ displaced blob (i.e. ①$'$) comes to rapid pressure equilibration with surroundings

i.e. $\boxed{\dfrac{\Delta z}{c_s} \ll \tau_{rise}}$ $\Leftrightarrow$ $\gamma < k c_s$
$\sim$ 'nearly incompressible'

$\therefore$

$$P_1' = P_1 + \Delta z \frac{dP_1}{dz} = P_2$$

so

$$P_1 \rho_1^{-\gamma} = P_1' \rho_1'^{-\gamma}$$

$$\Rightarrow \quad P_1 \rho_1^{-\gamma} = \left(P_1 + \Delta z \frac{dP_1}{dz}\right) \rho_1'^{-\gamma}$$

$$\Rightarrow \quad \left(\frac{\rho_1'}{\rho_1}\right)^{\gamma} = 1 + \frac{\Delta z}{P_1} \frac{dP_1}{dz}$$

$$\frac{\rho_1'}{\rho_1} = \left(1 + \frac{\Delta z}{P_1}\frac{dP_1}{dz}\right)^{1/\gamma} \cong 1 + \frac{\Delta z}{\gamma P_1} \frac{dP_1}{dz}$$

$$\frac{\rho_2}{\rho_1} = 1 + \frac{1}{\rho_1} \Delta z \frac{d\rho_1}{dz}$$

$\Rightarrow$ buoyant blob if:

$$\frac{\rho_1'}{\rho_1} < \frac{\rho_2}{\rho_1} \quad \Rightarrow \quad \frac{\Delta z}{\gamma} \frac{1}{P_1} \frac{dP_1}{dz} < \frac{\Delta z}{\rho_1} \frac{d\rho_1}{dz}$$

$$\Rightarrow \quad \boxed{\frac{1}{\gamma} \frac{1}{P_1} \frac{dP_1}{dz} < \frac{1}{\rho_1} \frac{d\rho_1}{dz}}$$

or, as both gradients negative

$$\boxed{\frac{1}{\gamma}\left|\frac{1}{P_1}\frac{dP_1}{dz}\right| > \frac{1}{\rho_1}\left|\frac{d\rho_1}{dz}\right|}$$

Schwarzchild criterion for Convective Instability

and as $S \equiv \ln(p\rho^{-\gamma})$

$$\frac{dS}{dz} = \frac{1}{p}\frac{dp}{dz} - \frac{\gamma}{\rho}\frac{d\rho}{dz}$$

$\Rightarrow$ blob buoyant if $\dfrac{dS}{dz} < 0 \rightarrow$ "superadiabatically stratified"

sinks/restored if $\dfrac{dS}{dz} > 0 \rightarrow$ "subadiabatically stratified"

Marginal $dS/dz = 0 \rightarrow$ adiabatically stratified

Note: $\rightarrow$ Schwarzschild instability criterion $\leftrightarrow$ answers "is free energy available, locally" $\leftrightarrow$ ideal

$\rightarrow$ Rayleigh # criterion $\Rightarrow$ $Ra > Ra)_{crit}$
$\Rightarrow$ does free energy overcome dissipation?

Now, what does δW say?

Recall: $$\delta W = \frac{1}{2}\int d^3x \left[\frac{\underline{Q}^2}{4\pi} + \gamma P (\underline{\nabla}\cdot\underline{\xi})^2 + \underline{J}_0\cdot(\underline{\xi}\times\underline{Q}) + (\underline{\xi}\cdot\underline{\nabla}P_0)(\underline{\nabla}\cdot\underline{\xi}) - (\underline{\xi}\cdot\underline{\nabla}\phi)\nabla\cdot(\rho_0\underline{\xi}) \right.$$

as pure hydro $\rightarrow$ $\underline{Q} = 0$, $\underline{J}_0 = 0$

$\dfrac{dp}{dz} = \rho g \rightarrow$ hydrostatic equilibrium
$\underline{\nabla}P = \cancel{\underline{J}\times\underline{B}} + \rho \underline{g}$

$\underline{g} = \nabla\phi$, $\underline{g}$ downward

$\therefore$

$$2\delta W = \int d^3x \Big[\gamma p\,(\nabla\cdot\underline{\xi})^2 + (\underline{\xi}\cdot\nabla p)(\nabla\cdot\underline{\xi}) + (\underline{\xi}\cdot\underline{g})\,(\underline{\xi}\cdot\nabla\rho_0 + \rho_0\,\nabla\cdot\underline{\xi}) \Big]$$

$$= \int d^3x \Big[\gamma p\,(\nabla\cdot\underline{\xi})^2 + (\nabla\cdot\underline{\xi})\big(\underline{\xi}\cdot(\nabla p + \underline{g}\,\rho_0)\big) + (\underline{\xi}\cdot\underline{g})\,(\underline{\xi}\cdot\nabla\rho_0) \Big]$$

but $\nabla p = \rho\,\underline{g}$ (eqbrm condition) $\Rightarrow$

$$2\delta W = \int d^3x \Big[\gamma p\Big((\nabla\cdot\underline{\xi})^2 + 2\frac{(\nabla\cdot\underline{\xi})(\underline{\xi}\cdot\nabla p)}{\gamma p} + \Big(\frac{\underline{\xi}\cdot\nabla p}{\gamma p}\Big)^2\Big) - \gamma p\Big(\frac{\underline{\xi}\cdot\nabla p}{\gamma p}\Big)^2 + (\underline{\xi}\cdot\underline{g})\,(\underline{\xi}\cdot\nabla\rho_0) \Big]$$

$$= \int d^3x \Big[\gamma p\Big(\nabla\cdot\underline{\xi} + \frac{\underline{\xi}\cdot\nabla p}{\gamma p}\Big)^2 - \frac{(\underline{\xi}\cdot\nabla p)^2}{\gamma p} + (\underline{\xi}\cdot\underline{g})\,(\underline{\xi}\cdot\nabla\rho_0) \Big]$$

$$= \int d^3x \Big[\gamma p\Big(\nabla\cdot\underline{\xi} + \frac{\underline{\xi}\cdot\nabla p}{\gamma p}\Big)^2 - \underline{\xi}\cdot\nabla p\Big(\frac{\underline{\xi}\cdot\nabla p}{\gamma p} - \underline{\xi}\cdot\frac{\nabla\rho_0}{\rho_0}\Big) \Big]$$

where used equilibrium condition again, so

$\Rightarrow$

$$2\delta W = \int d^3x \left[\gamma P \left(\underline{\nabla} \cdot \underline{\xi} + \frac{\underline{\xi} \cdot \underline{\nabla} P}{\gamma P} \right)^2 - \frac{\underline{\xi} \cdot \underline{\nabla} P}{\gamma} \, \underline{\xi} \cdot \underline{\nabla} \, \ln\left(P \rho^{-\gamma} \right) \right]$$

Now, object is to

→ explore possible displacements to see if $\delta W < 0$ possible

→ uncover any general condition

Now, expect $\underline{\xi}$ to have form:

$$\underline{\xi} = \mathrm{re}\left[\hat{\underline{\xi}}(z) e^{ikx} \right] \qquad \text{(must be real)}$$

so can choose $\underline{\nabla} \cdot \underline{\xi} = - \frac{\underline{\xi} \cdot \underline{\nabla} P}{\gamma P}$

→ equivalent to setting a relation between ξ_x, ξ_z.

$$\rightarrow \quad \underline{\nabla} \cdot \underline{\xi} \sim \frac{\xi}{\gamma P} \frac{dP}{dz} \sim \frac{\xi}{\gamma L_p}$$

↳ pressure scale height

so $\frac{|\nabla \cdot \xi|}{|\xi|} \sim 1/L_p$ → "weakly compressible", in accord with physical argument contrast $\frac{|\nabla\cdot\xi|}{|\xi|} \sim |k|$ → "strongly compressible" limit

so $2\delta W = -\int d^3x \left[\frac{\underline{\xi}\cdot\underline{\nabla}p}{\gamma} \; \xi\cdot\nabla \ln(P\rho^{-\gamma}) \right]$

$\frac{dP}{dz}\hat{z} = \underline{\nabla}P$ and $\frac{dP}{dz} < 0$ ⇒

if have any range of z over which

$$\frac{d}{dz}\ln(P\rho^{-\gamma}) < 0$$

⇒ have $\underline{\xi} \neq 0$ there, and $\delta W < 0$

⇒ instability, with criterion/condition that

$\boxed{\frac{d\ln(P\rho^{-\gamma})}{dz} < 0}$ → Schwarzschild Condition recovered

Now can go furthur, and ask what is effect of magnetic field?

i.e. — consider $\underline{B} = B_0 \hat{x}$

then
$$\delta W = \underbrace{\delta W_0}_{\text{what we have}} + \int d^3x \, \frac{Q^2}{8\pi}$$

$$\underline{Q} = \underline{\nabla} \times (\underline{\xi} \times \underline{B_0})$$

$$\underline{Q} = \underline{B_0} \cdot \underline{\nabla}\, \underline{\xi} - \overbrace{\cancel{\underline{\xi} \cdot \underline{\nabla}\, \underline{B_0}}}^{0 \text{ (homogeneous)}} - \underline{B_0}\, \underline{\nabla} \cdot \underline{\xi}$$

Now, to minimize δW, $\underline{B_0} \cdot \underline{\nabla}\, \underline{\xi} = 0$

→ flute displacement $k_\parallel = 0$

→ no bending energy expended

i.e. $\underline{Q} = -\underline{B_0}\, \underline{\nabla} \cdot \underline{\xi}$

$$\delta W = \delta W_0 + \int d^3x \, \frac{B_0^2}{8\pi} (\underline{\nabla} \cdot \underline{\xi})^2$$

but from before have, $\underline{\nabla} \cdot \underline{\xi} = -\dfrac{\underline{\xi} \cdot \underline{\nabla} p}{\gamma p}$

$$\delta W = \int d^3x \left[\frac{B_0^2}{8\pi} \left(\frac{\underline{\xi} \cdot \nabla p}{\gamma p} \right)^2 - \left(\frac{\underline{\xi} \cdot \underline{\nabla} p}{\gamma} \right) \frac{\underline{\xi}}{2} \cdot \nabla \ln (p \rho^{-\gamma}) \right]$$

$$\delta W \sim \int d^3x \left[P_{mag} \frac{\mathcal{E}^2}{\gamma^2 L_p^2} - \frac{P_{Th}}{\gamma L_p} \mathcal{E}^2 \left| \frac{dS}{dz} \right| \right]$$

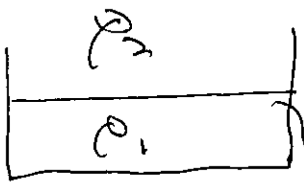
$\delta W < 0$ if $\quad \left| \frac{dS}{dz} \right| > \frac{P_{mag}}{P_{Th}\, \gamma L_p}$

$$\Rightarrow \frac{dS}{dz} < \frac{1}{\gamma \beta} \left(\frac{dP}{P\,dz} \right)$$

$\therefore$ indicates → magnetic field stabilizing
→ need critical entropy gradient $\sim \frac{1}{\beta L_p}$ for instability.

Moral of the story:
→ energy principle recovers essential physical criterion (Schwarzschild)
→ enables simple, quick, albeit imprecise insights into more complicated stability problems.

b.) Rayleigh-Taylor Instability → critical to implosions (ICF)



→ interface

$\downarrow g$ $\qquad \rho_2 > \rho_1$ $\qquad p = 0$ (cold) (as will $\nabla \cdot \underline{V} = 0$)

→ while nominally at equilibrium, configuration is unstable (heavy "falls" into light)

→ 

→ ripples, "spike and bubble"

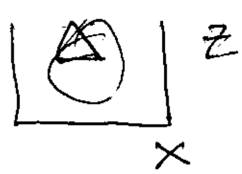
$$\gamma^2 = |kg| \frac{(\rho_2 - \rho_1)}{(\rho_2 + \rho_1)}$$

→ here $\nabla \cdot \underline{V} = 0$

→ if continuous profile $\qquad \downarrow g \qquad \nabla \rho \uparrow$

$$\frac{\partial \underline{\tilde{V}}}{\partial t} = -\frac{\nabla \tilde{p}}{\rho_0} - g \frac{\tilde{\rho}}{\rho_0} \hat{z}$$

$g > 0$



$$\frac{\partial}{\partial t} (\nabla \times \underline{\tilde{V}}) \cdot \hat{y} = 0 - g \nabla \times \left(\frac{\tilde{\rho}}{\rho_0} \hat{z} \right)$$

$$\underline{V} = -\partial_z \phi \, \hat{x} + \partial_x \phi \, \hat{z}$$

$$\nabla \cdot \underline{V} = 0$$

$$-\frac{\partial}{\partial t}\nabla^2\phi = g\,\partial_x\left(\frac{\tilde{\rho}}{\rho_0}\right)$$

$$\frac{\partial\tilde{\rho}}{\partial t} = -\partial_x\tilde{\phi}\frac{d\rho_0}{dz} \quad \Rightarrow \quad \omega^2 = -\frac{k_x^2}{k^2}\frac{g}{L_\rho}$$

$$\boxed{\gamma^2 = \frac{k_x^2}{k^2}\,g/L_\rho} \qquad \begin{matrix} g>0 \\ 1/L_\rho>0 \end{matrix}$$

interchange structure

Now, what would δW say?

$$\delta W = \frac{1}{2}\int d^3x\left[\frac{Q^2}{4\pi} + \gamma P(\nabla\cdot\underline{\xi})^2 + \underline{J}_0\cdot(\underline{\xi}\times\underline{Q})\right.$$

$$\left. + (\underline{\xi}\cdot\nabla P_0)(\nabla\cdot\underline{\xi}) - (\underline{\xi}\cdot\nabla\phi)(\nabla\cdot(\rho_0\underline{\xi}))\right]$$

$$\underline{Q}=0,\quad \underline{J}=0,\quad P=0,\quad \nabla\cdot\underline{\xi}=0$$

$$2\delta W = \int d^3x\left[-(\underline{\xi}\cdot\nabla\phi)(\rho_0\nabla\cdot\underline{\xi} + \underline{\xi}\cdot\nabla\rho_0)\right]$$

$$= \int d^3x\left[+(\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\nabla\rho_0)\right]$$

$$\delta W = \int\frac{d^3x}{2}\left[(\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\nabla\rho_0)\right]$$

$$\delta W = \int \frac{d^3x}{2} \left[(\underline{\xi} \cdot \underline{g})(\underline{\xi} \cdot \underline{\nabla} \rho_0) \right]$$

$g < 0$ so if $\underline{\nabla} \rho_0 > 0$ $\left(d\rho_0/dz > 0 \right)$ anywhere

$\Rightarrow \delta W < 0 \rightarrow$ instability

Now, if equilibrium hydrostatic:

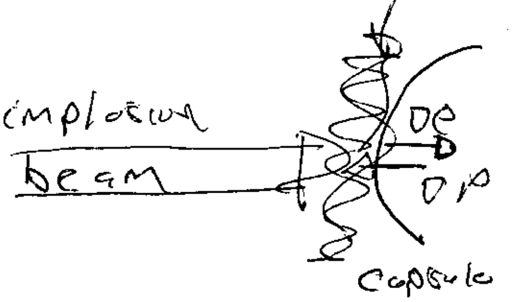
$$\underline{\nabla} p = \rho \underline{g} \;, \quad \Rightarrow$$

$$\boxed{\delta W = \int \frac{d^3x}{2} \left[(\underline{\xi} \cdot \nabla p) \left(\frac{\underline{\xi} \cdot \underline{\nabla} \rho_0}{\rho_0} \right) \right]}$$

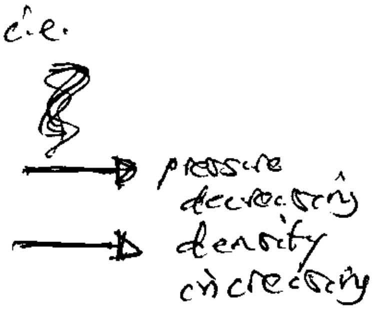
$\Rightarrow$ Rayleigh Taylor instability will result whenever $(\nabla p)(\nabla \rho) < 0$

$\rightarrow$ pressure density gradients opposite.
(i.e. pressure highest at bottom)

i.e. implosion beam



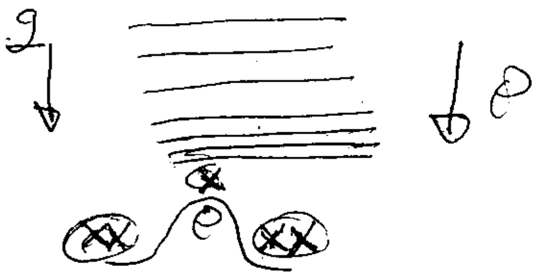
i.e.



iii) Interchange Instability

(basic confinement consideration)

→ consider plasma confined by magnetic pressure gradient



$$\nabla p = \underline{J}\times\underline{B} + \rho \underline{g}$$

str. lines

$$\frac{dp}{dz} = -\nabla\left(\frac{B^2}{8\pi}\right) + \cancel{\frac{\underline{B}\cdot\nabla\underline{B}}{4\pi}} + \rho\underline{g}$$

$\beta \ll 1$ $\quad$ $$\boxed{\nabla\left(\frac{B^2}{8\pi}\right) = \rho \underline{g}}$$ $\quad$ $\underline{g} = -g\hat{z}$, $p \to 0$

equilibrium

→ $$\delta W = \int d^3x \left[\frac{Q^2}{8\pi} + (\nabla\cdot\underline{\xi})^2 \gamma p_0 + \underline{j_0}\cdot(\underline{\xi}\times\underline{Q}) + (\underline{\xi}\cdot\nabla p_0)(\nabla\cdot\underline{\xi}) - (\underline{\xi}\cdot\nabla\phi)\,\nabla\cdot(\rho_0\underline{\xi}) \right]$$

$j_0 = 0$

$p_0 = 0$

$$\delta W = \int d^3x \left[\frac{Q^2}{8\pi} + (\underline{\xi}\cdot\underline{g})\left(\underline{\xi}\cdot\nabla\rho_0 + \rho_0 \nabla\cdot\underline{\xi}\right) \right]$$

Here, must address $\underline{Q}$,

$$\underline{Q} = \underline{B}_0 \cdot \underline{\nabla}\, \underline{\xi} - \underline{\xi} \cdot \underline{\nabla}\, B - B_0\, \underline{\nabla} \cdot \underline{\xi}$$

Now, can have $\underline{Q} = 0$ if:

$\rightarrow$ $\underline{B}_0 \cdot \underline{\nabla}\, \xi = 0$ i.e. $\underline{\xi}$ constant along $\underline{B}_0$

$\Rightarrow k_{\parallel} = 0$

and

$$\rightarrow \quad \underline{\nabla} \cdot \underline{\xi} = -\frac{\underline{\xi} \cdot \underline{\nabla} B_0}{B_0}$$

$\therefore$

$$\delta W = \int d^3x \left[(\underline{\xi} \cdot \underline{g})\, \rho_0 \left(\frac{\underline{\xi} \cdot \underline{\nabla}\, \rho_0}{\rho_0} - \frac{\underline{\xi} \cdot \underline{\nabla} B_0}{B_0} \right) \right.$$

$$= \int d^3x \left[(\underline{\xi} \cdot \underline{g}\, \rho_0)\; \underline{\xi} \cdot \underline{\nabla} \ln (\rho / B) \right]$$

$g < 0 \Rightarrow$ if $\nabla \ln (\rho / B) > 0$ anywhere

$\therefore$ instability there!

Now:

→ obvious parallel to Rayleigh-Taylor is

$\nabla \rho > 0 \iff \nabla \ln(P/B) > 0$

→ as $k_{\parallel} = 0$, field lines not bent

so can think of instability motion as interchange of flux tubes

Key question: Does interchange lower/raise potential energy?

interchange conserves magnetic flux

$$\phi_2 = \int B_2 \, da = B_2 A_2$$

$$\phi_1 = \int B_1 \, da = B_1 A_1$$

$M \Leftrightarrow m/\text{length}$

so

$$M_2 = \left(\frac{\rho}{B}\right)_2 \Phi_2$$

$$M_1 = \left(\frac{\rho}{B}\right)_1 \Phi_1$$

but $\phi_1 = \phi_2 \Rightarrow$

$$M_{2,1} = (P/B)_{2,1} \, \Phi$$

so $\quad \nabla M \gtrless 0 \Rightarrow \nabla (P/B) \gtrless 0$

$\Rightarrow$ if P/B increases interchange will liberate gravitational potential energy, i.e. instability, a la' R.-T.

$\rightarrow$ why care?:

- (interchange instability) severely degrades plasma confinement
- curing interchange stability is key element in device design $\rightarrow$ "minimum-B"

iv.) Interchange without Gravity

- in the context of magnetic confinement, "g" is a crutch, to represent curved field lines

– i.e. 

$$\underline{a} = \frac{V^2}{R_c} \rightarrow \underline{g}_{eff}$$

[centrifugal acceleration on particle]

– natural to investigate interchanges *without* "$\underline{g}$" ⇒ pressure gradient drive (expansion free energy)

– now

$$\delta W = \int d^3x \left[\frac{\underline{Q}^2}{8\pi} + \gamma p (\underline{\nabla} \cdot \underline{\xi})^2 + \underline{\xi} \cdot \underline{\nabla} p \, (\underline{\nabla} \cdot \underline{\xi}) + \underline{j} \cdot \underline{\xi} \times \underline{Q} \right]$$

Now, $\underline{Q} = 0$ → avoid bending, etc.

$$\underline{\nabla} \times (\underline{\xi} \times \underline{B}_0) = 0$$

$$\Rightarrow \quad \underline{\xi} \times \underline{B}_0 = \nabla \phi$$

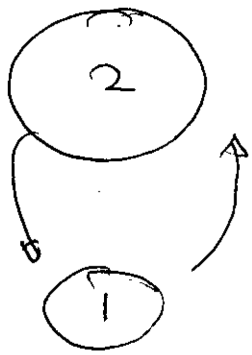
↳ some scalar potential

and $\underline{B} \cdot \underline{\nabla} \phi = 0$. ⇒ ϕ constant along lines of force ...

and can formulate dW in terms ϕ, or....

$\Rightarrow$ consider interchange, with flux conservation

$$\Phi_1 = \Phi_2$$



Does interchange raise or lower energy?

$$\Delta E = [\text{final energy of } ①] - [\text{initial energy of } ①]$$
$$+ [\text{final energy of } ②] - [\text{initial energy of } ②].$$

where interchange

a) "puts" ① into ② slot
"puts" ② into ① slot

b) keeps $p\rho^{-\gamma} = pV^{\gamma} = \text{const}$

$V \equiv$ volume of flux tube

$\Rightarrow$ final energy of ① $\rightarrow$ $(\text{new } p)\, V_2/(\gamma-1)$ $\hookrightarrow p_1'$

final energy of ② $\rightarrow$ $(\text{new } p)\, V_1/\gamma-1$ $\hookrightarrow p_2'$

so ...

$$\Delta E = \Delta W = \frac{1}{(\gamma-1)}\left[\left(p_1' V_2 - p_1 V_1\right) + \left(p_2' V_1 - p_2 V_2\right)\right]$$

and

$$\left.\begin{aligned} p_1' V_2^{\gamma} &= p_1 V_1^{\gamma} \\ p_2' V_1^{\gamma} &= p_2 V_2^{\gamma} \end{aligned}\right\}$$

from eqn. state
$p' \equiv$ pressures of displaced flux tubes
(argument akin to Schwarzschild)

$\Rightarrow$

$$(\gamma-1)\,\Delta W = \left\{ p_1\left[\left(\frac{V_1}{V_2}\right)^{\gamma} V_2 - V_1\right] + p_2\left[\left(\frac{V_2}{V_1}\right)^{\gamma} V_1 - V_2\right]\right\}$$

$$V_2 = V_1 + \delta V$$
$$p_2 = p_1 + \delta p$$

$$(\Delta W)(\gamma-1) = \left\{ p_1\left[\left(\frac{V_1}{V_1+\delta V}\right)^{\gamma}(V_1+\delta V) - V_1\right] + (p_1+\delta p)\left[\left(\frac{V_1+\delta V}{V_1}\right)^{\gamma} V_1 - (V_1+\delta V)\right]\right\}$$

$$(\gamma-1)\,\Delta W = \Big\{ P_1 V_1 \Big[\Big(1+\frac{\delta V}{V}\Big)^{-(\gamma-1)} - 1 \Big]$$

$$+ P_1 V_1 \Big(1+\frac{\delta p}{p}\Big)\Big[\Big(1+\frac{\delta V}{V}\Big)^{\gamma} - \Big(1+\frac{\delta V}{V}\Big)\Big]\Big\}$$

$$= P_1 V_1 \Big\{ \Big[\cancel{1} - (\gamma-1)\frac{\delta V}{V} + \frac{(\gamma-1)(\gamma)}{2}\Big(\frac{\delta V}{V}\Big)^2 - \cancel{1} \Big]$$

$$+ \Big(1+\frac{\delta p}{p}\Big)\Big[\cancel{1} + \gamma\frac{\delta V}{V} + \gamma\frac{(\gamma-1)}{2}\Big(\frac{\delta V}{V}\Big)^2 - \cancel{1} - \frac{\delta V}{V}\Big]\Big\}$$

$$= P_1 V_1 \Big\{ \cancel{-(\gamma-1)\frac{\delta V}{V}} + \frac{(\gamma-1)(\gamma)}{2}\Big(\frac{\delta V}{V}\Big)^2$$

$$+ \cancel{\gamma\frac{\delta V}{V}} - \cancel{\frac{\delta V}{V}} + \frac{\delta p}{p}(\gamma-1)\frac{\delta V}{V} + \gamma\frac{(\gamma-1)}{2}\Big(\frac{\delta V}{V}\Big)^2 \Big\}$$

$\therefore$

$$\boxed{\frac{\Delta W}{P_1 V_1} = \gamma\Big(\frac{\delta V}{V}\Big)^2 + \frac{\delta p}{p}\frac{\delta V}{V}}$$

$\Rightarrow$ generic expression for interchange δW

>0 $\qquad$ >0 or <0

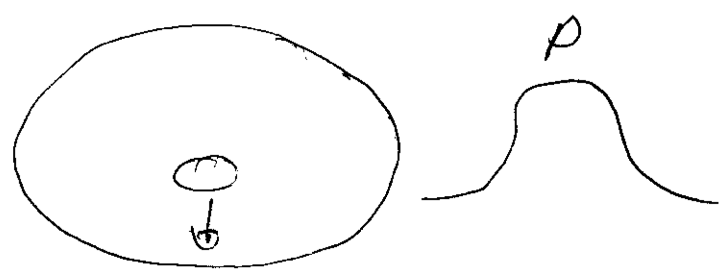
→ clearly,

$$\frac{\delta V}{V} \sim (\nabla \cdot \underline{\xi}) \quad , \quad \frac{\delta p}{p} \sim \quad \underline{\xi} \cdot \nabla p$$

and

→ expansion free energy relaxation ⇒

$\underline{\delta p < 0}$ → i.e.



pressure higher in center so $\delta p < 0$ ⇒ relaxation occurs.

$$\therefore \text{ key is sign } \frac{\delta V}{V} \begin{cases} > 0 \rightarrow \text{instability} \\ < 0 \rightarrow \text{stability} \end{cases}$$

→ Now, for flute perturbation $(k_{\parallel} = 0)$

$$V = \int S \, d\ell$$

$S \equiv$ cross-sectional area of tube.

but $\Phi = B(\ell)\, S(\ell) = \text{const}$

⇒

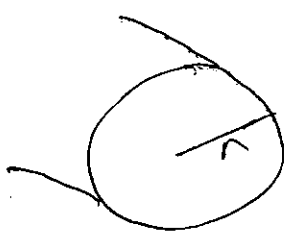
$$V = \Phi \int \frac{dl}{B} \quad \Rightarrow \quad \frac{\delta V}{V} < 0$$

⇒ $$\boxed{\delta \int \frac{dl}{B} < 0}$$ → condition for interchange stability

$$\underbrace{\delta p}_{<0} \; \underbrace{\delta V}_{<0} > 0$$

→ content of criterion is that configuration should have a minimum in B in the core, to confine pressure

i.e.



then stable if:

$$\delta \int \frac{dl}{B} < 0 .$$

⇒ "minimum B" criterion for stability.

→ if define ψ → label of surface enclosing const flux Φ

∴ $V(\psi) \equiv$ volume enclosed by flux surface

$p(\psi) \equiv$ pressure enclosed

$$dp/d\psi < 0 \quad \Rightarrow \quad \text{need} \quad \frac{d^2V}{d\psi^2} > 0$$

$\Longleftrightarrow$ minimum-B

→ Can re-write instability criterion

$$\delta W = p_1\, \delta V \left(\gamma \frac{\delta V}{V_1} + \frac{\delta p}{p_1} \right)$$

$$= p_1\, \delta V \left[\delta \ln (p V^\gamma) \right]$$

so $\delta(pV^\gamma) < 0$ → inst. (akin Schwarzschild)

Also, if tube ① has flux ψ, then

$$V = u\,\psi$$

$\Rightarrow$

$$\frac{\delta W}{\psi} = p\,\delta u\,\frac{\delta(pu^{\gamma})}{pu^{\gamma}} < 0$$

→ What does it Mean ?

$$\underset{\text{volume}}{V} = \int dl\, A = \Phi \int \frac{dl}{B}$$

now ∇p → "expansion free energy"

$\delta V > 0 \Rightarrow \delta \int \frac{dl}{B} > 0$ → fluid element expands
⇒ tends reduce W_p

$\delta V < 0 \Rightarrow \delta \int \frac{dl}{B} < 0$ → fluid element compresses
⇒ tends increase W_p

$\delta V > 0$ → 'maximum B'
$\delta V < 0$ → 'minimum B'

Can then define :

$$\underset{\text{potential energy of tube}}{E_p} = -p\, U \quad , \quad U = -\oint \frac{dl}{B}$$

∴ → can argue tube tends to move in direction of lower U.
→ equilibrium for $p = p(U)$

then, not surprisingly, can develop parallel between convection and interchange

i.e.

Convection	Interchange
gravitational potential energy	$E_p \to$ expansion energy
blob	flux tube
displace blob	displace tube
$\rho' < \rho_{ambient}$ $\Leftrightarrow$ buoyant rise	$p' < p_{ambient}$ $\Leftrightarrow$ expansion continues $\left(dp/dU < 0 \right)$ (squeezed out)
adiabatic profile $\frac{\nabla p}{p} = \gamma \frac{\nabla \rho}{\rho}$	adiabatic displacement $-\gamma p \frac{\delta U}{U} = \frac{dp}{dU} \delta U$
Schwarzschild Criterion $\frac{\nabla p}{p} < \gamma \frac{\nabla \rho}{\rho}$ for instability	Interchange Criterion $\frac{dp}{dU} \delta U > -\gamma p \frac{\delta U}{U}$ $\Rightarrow \frac{dp}{dU} > -\frac{\gamma p}{U}$

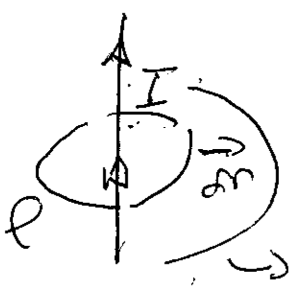
∴ for instability: $\underbrace{\frac{dp}{du}}_{\text{change from relaxation}} > \underbrace{\frac{-\gamma p}{u}}_{\text{adiabatic pressure change}}$

for stability, need:

$$\boxed{\frac{dp}{du} < \frac{\gamma p}{|u|}}$$

→ Consider some configurations (magnetic)

a.) single wire



- stability to displacement δr
- $\nabla p/p$ limit ?

now $\delta \int \frac{d\ell}{B}$

$d\ell = 2\pi r$

$B = 2I/r$

$d\ell/B \sim \frac{\pi}{I} r^2$ → wire is "minimum-B" !

for δp limit: $\quad \dfrac{dp}{dU} < \dfrac{\gamma p}{|U|}$

$$U = -\int \frac{dl}{B} \sim -r^2$$

$$\frac{dp}{dU} = \frac{dp}{dr}\frac{dr}{dU}$$

$$= \frac{dp}{dr}\left(\frac{1}{2r}\right) \quad \Rightarrow \left|\frac{1}{p}\frac{dp}{dr}\right| < \frac{\gamma(2r)}{r^2}$$

$$\therefore \left|\frac{1}{p}\frac{dp}{dr}\right| < \frac{2\gamma}{r} \quad \Rightarrow \boxed{\left|\frac{d\ln p}{d\ln r}\right| < 2\gamma}$$

$\Rightarrow$ imposes limit on pressure gradient for interchange stability. $\Rightarrow$ "β limits"

ii)
can approach point dipole similarly $\rightarrow$ i.e. earth.
i.e. $\quad B \sim 1/r^3$

$$dl \sim r \quad \Rightarrow \quad U \sim -r^4$$

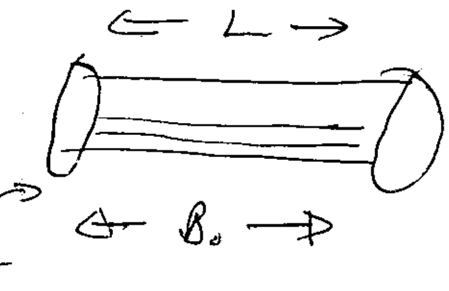
similar reasoning $\Rightarrow -\dfrac{d\ln p}{d\ln r} < 4\gamma$

→ Line Tying and Conducting End Plates

- Till now, have ignored boundary

⇒ consider plasma between two conducting endplates

i.e.

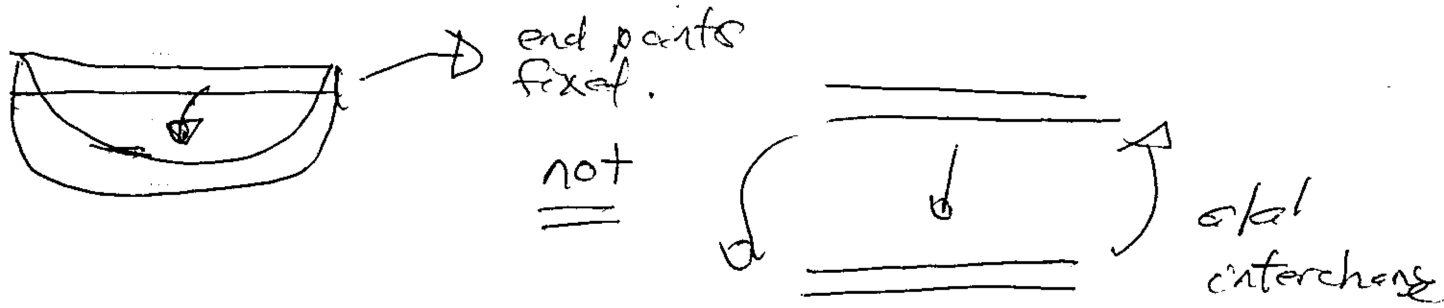


$\underline{E}_t = 0$ on plate

$$\Rightarrow \underline{\mathcal{E}}\Big|_{plate} = 0$$

lines are tied

i.e. displacement has form:



⇒ field lines bent!

now, $\underline{U}_0 = 0$ ⇒

$$\delta W = \int d^3x \left[\frac{\underline{Q}^2}{8\pi} + \gamma p (\underline{\nabla} \cdot \underline{\xi})^2 + (\underline{\xi} \cdot \underline{\nabla} p_0)(\underline{\nabla} \cdot \underline{\xi}) \right]$$

$$\underline{Q} = \underline{\nabla} \times \underline{\xi} \times \underline{B_0}$$

$$= \underline{B_0} \cdot \underline{\nabla} \underline{\xi} - \underline{\xi} \cdot \underline{\nabla} \underline{B_0} - \underline{B_0}\, \underline{\nabla} \cdot \underline{\xi}$$

$$\underline{\nabla} \cdot \underline{\xi} \neq 0$$

new stabilizing effect

$$\delta W = \int d^3x \left[\frac{(\underline{B_0} \cdot \underline{\nabla} \underline{\xi} - \underline{B_0}\, \underline{\nabla} \cdot \underline{\xi})^2}{8\pi} + \gamma p (\underline{\nabla} \cdot \underline{\xi})^2 + (\underline{\xi} \cdot \underline{\nabla} p_0)\, \underline{\nabla} \cdot \underline{\xi} \right]$$

i.e. can't take $\underline{B_0} \cdot \underline{\nabla}\, \underline{\xi} = 0$ any more

so $\quad Q \sim B_0 \dfrac{\partial \xi_r}{\partial z}$ $\quad$ i.e. can make $(\underline{\nabla} \cdot \underline{\xi})\, \underline{B_0}$ smaller ..

$$\delta W \sim V \left[\frac{B_0^2}{8\pi} \left(\frac{\partial \xi_r}{\partial z} \right)^2 + \gamma p \left(\frac{\delta V}{V} \right)^2 + \delta p \frac{\delta V}{V} \right]$$

i.e. schematic ....

$$\frac{\partial \xi_r}{\partial z} \sim \frac{\xi_r}{L} \qquad \frac{\delta V}{V} = \frac{\nabla V}{V} \xi_r$$

$$\delta p = \nabla p\, \xi_r$$

$\Rightarrow$

$$\delta W \sim V\left\{\left(\frac{B_0^2}{8\pi L^2} + \gamma p\left(\frac{\nabla U}{U}\right)^2 + \nabla p \frac{\nabla U}{U}\right)\varepsilon^2\right)$$

$\therefore$ $\delta W < 0 \rightarrow$ instability $\Rightarrow$

instability if $\quad -\nabla p \frac{\nabla U}{U} < \gamma p\left(\frac{\nabla U}{U}\right)^2 + \underbrace{\frac{B^2}{8\pi L^2}}_{\substack{\text{additional} \\ \text{stabilizing} \\ \text{effect}}}$

$\Rightarrow$ line tying raises
critical pressure gradient

$\Rightarrow$ clearly stabilizing $\Rightarrow$ β limit!

Physics $\rightarrow$ fixing end points forces
bending of field lines

$\rightarrow$ lose interchange structure

$\rightarrow$ energy expended coupling to
plucking magnetic field lines.